KODAIRA DIMENSION OF ALGEBRAIC FUNCTION FIELDS

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0. Introduction. The concept of *Kodaira dimension*, together with Iitaka's *fibering theorem* (see [I1] and [I2]), is one of the most fundamental tools in the theory of birational classification of algebraic varieties. A general theory in this respect, when the base field is of characteristic 0, has been developed by Iitaka and his school since the early seventies. The object of this work is to extend some of their results to the cases of positive characteristic. We adopt an algebro-arithmetic approach, which has the advantage that most of the discussions are characteristic free. We shall define the Kodaira dimension for any algebraic function field, and shall prove a "fibering theorem" for the Kodaira dimension of algebraic function fields.

Let K/k be an algebraic function field of $\dim K/k = n > 0$ with the characteristic $\mathrm{ch}\, K/k = p \geq 0$. Thus K is a finitely generated extension over the base field k which is assumed to be algebraically closed in K, and trans.deg $K/k = n$. For the time being we assume that K/k has a nonsingular complete model X, i.e. X is a nonsingular complete variety defined over k and K is the rational function field of X. The canonical ring $C(X)$ of X is a k-graded integral closed algebra defined by

$$C(X) = \bigoplus_{i \geq 0} H^0(X, \mathcal{O}_X(i\omega_X))$$

where ω_X is a canonical divisor of X. It is well known that $C(X)$ is a birational invariant of X, i.e. X is uniquely determined by the algebraic function field K/k, up to a k-graded algebra isomorphism. Thus it is legitimate to write $C(K/k)$ for $C(X)$, and call $C(K/k)$ the *canonical ring* of K/k.

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From $C(K/k)$ we derive the other "canonical invariants" of K/k, such as the *Kodaira dimension* $\kappa(K/k)$ of K/k, given by the formula

$$\kappa(K/k) = \text{trans.deg } C(K/k)/k - 1,$$

and the *canonical field* $Z(K/k)$ of K/k, which is a subfield of K generated by the quotients of homogeneous elements of $C(K/k)$ with the same degree. The canonical field of K/k is algebraically closed in K, and if $\kappa(K/k) \neq -1$, then $\kappa(K/k) = \text{trans.deg } Z(K/k)/k$.

We now assume $0 \leq \kappa(K/k) < n$, in which case $K/Z(K/k)$ is an algebraic function field of dimension $n - \kappa(K/k) > 0$ with

$$X' = \{x \in X \mid \text{any rational function } f \text{ of } X \text{ contained in } Z(K/k) \text{ is locally regular at } x\}$$

as a nonsingular model. Suppose X' is complete. Then the canonical invariants $C(K/Z(K/k))$, $\kappa(K/Z(K/k))$ and $Z(K/Z(K/k))$ are defined for $K/Z(K/k)$. We shall prove that the Kodaira dimension of $K/Z(K/k)$ *always vanishes*. This is the "fibering theorem" for the Kodaira dimension of algebraic function fields.

We now describe the structure of the paper. Hoping to prove the "fibering theorem" for any algebraic function field, we drop the assumption that K/k has a nonsingular complete model. A large part of the paper is devoted to constructing the canonical ring $C(K/k)$ of K/k, using absolute differentials of K over its prime field. Let $D(K)$ be the absolute differential module of K over its prime field. A subset B of K is called a differential basis of K/k if $dB = \{db \mid b \in B\}$ is a K-basis of $D(K)$ and $B - B \cap k$ is a finite set. Let R be a regular locality of K/k (i.e. R is the local ring of a regular point of an affine model of K/k). We shall prove that the R-module RdR which is generated by dR in $D(K)$ over R is a free R-module with a R-basis dB, where B is a differential basis of K/k contained in R (see Corollary 2.7). We call B a set of *uniformizing coordinates* of R.

In Section 3 we shall define, for any two differential bases B_1, B_2 of K/k, an element $J(B_1, B_2) \in K^*/k^*$, where K^* and k^* are the multiplicative groups of K and k respectively, such that, if B_1 and B_2 are two sets of uniformizing coordinates of a regular locality R of K/k, then $J(B_1, B_2)$ is the residue class of an invertible element x of R in K^*/k^*. $J(B_1, B_2)$ is called the "generalized Jacobian" of B_1 and B_2.

In Section 4 we define the canonical ring of K/k. Let $A = \bigoplus_{i \geq 0} A_i = K[X]$ be a polynomial ring in one variable X over K, considered as a k-graded algebra with A_i as the i-th homogeneous component. We can find a map τ from the set of differential bases of K/k to A_1, such that, for any two differential bases B_1 and B_2 of K/k, we have $\tau(B_1) = a\tau(B_2)$, where $a \in K$ with the residue class $a^* = J(B_1, B_2)$. Let R be any regular locality of K/k and B_R a set of uniformizing coordinates of R. The polynomial ring generated by the element $\tau(B_R)$ in A over R, denoted by $\tau(R)$, is independent of the choice of B_R. We define the canonical ring $C(K/k)$ of K/k to be the graded k-subalgebra $\bigcap \tau(R)$ of A, where R runs through the set of all regular localities of K/k.

The Kodaira dimension $\kappa(K/k)$ of K/k is then defined by the same formula as before, and the canonical field $Z(K/k)$ of K/k can be obtained as the subfield

$$K \cap QC(K/k)$$

of K, where $QC(K/k)$ is the quotient field of $C(K/k)$ as a subfield of QA.

If K/k has a nonsingular complete model X with a canonical divisor ω_X, we shall prove (see Theorem 4.7) that $C(K/k)$ defined above is isomorphic to $C(X) = \bigoplus_{i \geq 0} H^0(X, \mathcal{O}_X(i\omega_X))$. In general, it is easy to show that $C(K/k)$ is uniquely determined by K/k, up to a k-graded algebra isomorphism.

We now describe the "fibering theorem" for Kodaira dimension of algebraic function fields:

Let K/k be an algebraic function field of dimension n with $0 \leq \kappa(K/k) < n$. Let $Z(K/k)$ be the canonical field of K/k. Then $K/Z(K/k)$ is an algebraic function field with the Kodaira dimension $\kappa(K/Z(K/K)) = 0$.

A proof of this theorem is given in Section 5, under the assumption that K/k has a normal complete model X of the same Kodaira dimension as K/k (e.g. X is nonsingular). Although this is apparently a very weak restriction, we do not know at present how to remove it.

In the first two sections we give a presentation of the theory of uniformizing coordinates of regular localities, following Zariski [Z1], Zariski and Falb [Z-F], and Kunz [K]. A notable feature of our approach is perhaps the systematic adoption of the language of differential bases of abstract fields. We hope this rather straightforward and elementary treatment may have interest of its own.

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1. Differential bases. Given a field K with a subfield k, we let $D(K/k)$ denote the differential module of K over k. A subset B of K is called *K/k-differentially independent*, if the subset $dB = \{db | b \in B\}$ of $D(K/k)$ is K-linearly independent. If dB is further a K-linear basis of $D(K/k)$, we say that B is a *K/k-differential basis*.

These concepts can be introduced in several other ways. First consider the following statements concerning a subset B of K:

(a) for every element x of B there exists a derivation D_x of K over k such that $D_x(x) = 1$ and $D_x(y) = 0$ for every element y of B different from x.

(b) any element x of B is not separably algebraic over the subfield $K^p k(B - \{x\})$ of K (here $K^p = \{y^p | y \in K\}$ for K of prime characteristic $p > 0$. If ch $K = 0$ we let K^P denote the prime field of K).

(c) if D is a derivation of K over k such that $D(B) = 0$, then $D(K) = 0$.

(d) the field K is separably algebraic over $K^p k(B)$.

(d′) the field K is separably algebraic over $k(B)$.

LEMMA 1.1. *The above condition* (a) *or* (b) (*resp.* (c) *or* (d)) *holds for a subset B of K if and only if B is K/k-differentially independent* (*resp. dB is a set of generators of $D(K/k)$*). *If K/k is finitely generated, then* (c) (*or* (d)) *and* (d′) *are equivalent.*

Proof. The assertions about (a) and (c) are easy consequences of the universal property of the canonical derivation $d: K \to D(K/k)$. The equivalence of (a) and (b) (also that of (c) and (d)) follows from the well known facts of extending derivations of fields (see [L], p. 268; the proof of the last statement of the lemma is on the same page). Q.E.D.

COROLLARY 1.2. (*see also* [M], *p.* 271). (a) *If K is a field of characteristic* 0, *then B is K/k-differentially independent if and only if B is algebraically independent over k. B is a K/k-differential basis if and only if B is a transcendence basis of K over k.*

(b) *If K is of prime characteristic $p > 0$, then B is K/k-differentially independent if and only if B is p-independent over k. B is a K/k-differential basis if and only if B is a p-basis of K over k.*

In this work we shall use the unified concept of a differential basis exclusively. Hence in the following a familiarity with the concept of a p-basis on the part of the readers is not required.

Our main concern of this section is the precise relationship between the differential bases of a field and those of its finitely generated extensions. For that purpose some notations are in order.

Let G and G' be two arbitrary sets. If the set $G - G \cap G'$ is finite, we denote this by $G < G'$. If $G < G'$ and $G' < G$, then we write $G \sim G'$. Obviously "$\sim$" is an equivalence relation. If $G \sim G'$, then $G - G \cap G'$ and $G' - G \cap G'$ are two finite sets with, say, $r \geq 0$ and $s \geq 0$ elements, respectively. The difference $r - s$ will be denoted by $|GG'|$. Clearly $|GG''| = |GG'| + |G'G''|$ for any $G \sim G' \sim G''$.

We start with the simple extensions:

LEMMA 1.3. *Let $K = F(x)$ be a simple extension over a field F containing a subfield k, and E a F/k-differential basis. Then*

(a) *If x is transcendental (resp. separably algebraic) over F, then $E \cup \{x\}$ (resp. E) is a K/k-differential basis.*

(b) *If x is purely inseparable over F, and if*

$$X^p - a \qquad (a \in F, p > 0)$$

is the irreducible polynomial satisfied by x, then:

(b1) *if $a \in F^p k$, $E \cup \{x\}$ is a K/k-differential basis.*

(b2) *if $a \notin F^p k$, there exists an element y of E such that $(E \cup \{x\}) - \{y\}$ is a K/k-differential basis.*

(b3) *if k is perfect, then only* (b2) *can happen.*

Proof. The assertions (a), (b1) and (b2) follow easily from Lemma 1.1. For part (b3), notice that $a \notin F^p$ (because otherwise $X^p - a$ would be reducible over F), and $F^p = F^p k$ since k is perfect. Q.E.D.

LEMMA 1.4. *Let K/F be a finitely generated extension with* trans.deg $K/F = r$, *and k a subfield of F.*

(a) *If B and B' are K/k-differential bases and $B > B'$, then $B \sim B'$ and $|BB'| = 0$.*

(b) *If E is a F/k differential basis, then there exists a K/k-differential basis B such that $E \sim B$.*

(c) *If B is a K/k-differential basis and $B < F$, then there exists a F/k-differential basis E such that $B \sim E$.*

(d) *There exists a unique integer $[KF]_k \geq 0$ such that, if B and E are differential bases of K and F over k respectively, and $B \sim E < F$, then $|BE| = r + [KF]_k$.*

(e) *If E is a F/k-differential basis, B a subset of K with $B \sim E$ and $|BE| = r + [KF]_k$ (see* (d)*), then B is K/k-differentially independent if and only if dB is a set of generators of $D(K/k)$. In either case B is a K/k-differential basis.*

(f) *If k is perfect, then $[KF]_k = 0$.*

Proof. Part (a) follows from the basic properties of the linear bases of a linear space. The assertion (b) holds for a simple extension, according to Lemma 1.3; by induction it also holds for any finitely generated extension.

To prove (c), let $G = B \cap F$. Then the assumption $B < F$ implies that $G \sim B$. Since G is K/k-differentially independent, it is also F/k-differentially independent (by Lemma 1.1). Let E be a F/k-differential basis containing G. According to (b) we may find a K/k-differential basis B' having the relation $E \sim B'$. Then we have $B \sim G < E \sim B'$, which, by part (a), implies that $B \sim B'$, because B and B' are both K/k-differential bases. Now we have $B \sim G < E \sim B' \sim B$. Hence $B \sim E$.

For part (d) we need to show that, for any K/k-differential basis B and any F/k-differential basis E with $B \sim E < F$, the integer $|BE| - r$ is nonnegative and is independent of the choices of B and E (then we may let $[KF]_k = |BE| - r$). Let $K = F(x_1, \ldots, x_n)$. We proceed by induction on n. For $n = 1$ this is true by (a) and Lemma 1.3. So we assume that (d) holds up to $n - 1$. Write $K' = F(x_1, \ldots, x_{n-1})$. Put $t =$ trans.deg $K/K' \leq 1$ and $s =$ trans.deg $K'/F \leq n - 1$ (note $s + t = r$). Let B and E be as in the lemma. According to part (b), there exists a K'/k-differential basis B' such that $B' \sim E$. Then $|BB'| = t + [KK']_k$ and $|B'E| = s + [K'F]_k$, where $[KK']_k$, $[K'F]_k \geq 0$ by the inductive assumption. Since

$$|BE| = |BB'| + |B'E| = s + t + [KK']_k + [K'F]_k$$
$$= r + [KK']_k + [K'F]_k,$$

we have $|BE| - r = [KK']_k + [K'F]_k$. The sum on the right side is, by the inductive assumption, a nonnegative integer, which is independent of the choices of B and E. This proves (d).

Part (e) follows directly from (d).

Finally, if k is perfect, Lemma 1.3, (b3) implies that $[KF]_k = 0$ if K/F is a simple extension. The general cases for (f) can be proved by induction. Q.E.D.

LEMMA 1.5. *Let K/F be a finitely generated extension of* trans.deg $K/F = r$ *and k a subfield of F.*

(1) *If K/F is separably generated with a separating transcendence basis N, and E a F/k-differential basis, then $B = E \cup N$ is a K/k-differential basis, and $[KF]_k = 0$;*

(2) *If $[KF]_F = 0$, then K/F is separably generated, and any K/F-differential basis B' is a separating transcendence basis of K/F.*

Proof. (1) The assertion that B is a K/k-differential basis comes from Lemma 1.3, (a) by induction. It implies that

$$[KF]_k = |BE| - r = |N| - r = 0$$

for any subfield k of F.

(2) Suppose $[KF]_F = 0$ and B' a K/F-differential basis. Then $|B'| = r + [KF]_F = r$. But K is separably algebraic over $F(B')$, in view of Lemma 1.1, (d′). Thus B' is a separating transcendence basis of K/F. Q.E.D.

COROLLARY 1.6. *K/F is a separably generated extension if and only if $[KF]_k = 0$ for any subfield k of F.*

2. Uniformizing coordinates. Throughout this section K/F will be a fixed algebraic function field of dimension n (i.e. K is a finitely generated extension over the ground field F; F is algebraically closed in K and trans.deg $K/F = n > 0$). By a locality of K/F we mean a subring R of K satisfying the following conditions:

(1) $F \subseteq R$;

(2) the quotient field QR of R is K;

(3) there exists a finitely generated algebra $A = F[x_1, \ldots, x_r]$ over F with a prime ideal P of A such that

$$R = A_P$$

where A_P is the quotient ring of A with respect to P. The maximal ideal of R is denoted by M and the residue field R/M of R by L. Passing to the residue classes, we habitually regard F as a subfield of L.

Definition 2.1. Let k be a subfield of F and B a K/k-differential basis contained in R. We say B is a set of *uniformizing coordinates* of R over k, if the R-module RdR is free with $dB = \{db \mid b \in B\}$ as a R-free basis (or equivalently, if $RdR = RdB$), here RdR is the R-module generated by dR in $D(K/k)$; if furthermore B contains a set of generators of the maximal ideal M of R, then B is called a set of *normal uniformizing coordinates* of R over k. A subset H of K is said to be *R-independent* over k if there exists a set of uniformizing coordinates of R over k containing H.

We fix some notations which will be used throughout this section:

$d: K \to D(K/k)$ is the canonical derivation of K over k.

$E^* = \{e^*_i\}$ is a L/k-differential basis with $E^* < F$.

$E = \{e_i\}$ is a lift of E^* to R such that the elements of $E^* \cap F$ are lifted back to themselves under the above identification of F (thus $E \sim E^*$).

B' is a F/k-differential basis with $B' \sim E$.

$B \subseteq R$ is a K/k-differential basis with $B \sim E$.

(See Lemma 1.4 for the existence of such $B \sim B' \sim E \sim E^*$.)

LEMMA 2.2. *For any $x \in R$, there exists an element $w \in M$, such that $dx = \Sigma\, a_i de_i + cdw$, where a_i, $c \in R$, $e_i \in E$ and $c \notin M$.*

Proof. Let x^* denote the residue class of x in L. By Lemma 1.1, (d), x^* has an irreducible separable polynomial $f^*(X)$ over the subfield $k(L^p(E^*))$ ($p = \text{ch } k$; see Lemma 1.1 (b) for the meaning of L^p). We may assume that $f^*(X)$ has coefficients in $k[L^p \cup E^*]$. Let $f(X) \in R[X]$ be a lift of $f^*(X)$ such that the elements of $L^p \cup E^*$ are lifted back to $(R \cap K^p) \cup E$, and the elements of k remain unchanged. Since $f^*(x) = 0$ and

$f^{*\prime}(X)|_x \neq 0$ (here $f^{*\prime}(X)$ is the result of differentiating $f^*(X)$ with respect to X), we have $f(x) \in M$ and $f'(x) \notin M$. Then

$$df(x) = \Sigma\, a_i de_i + f'(x)dx;$$

$$dx = (f'(x))^{-1}df(x) + \Sigma\, (-f'(x))^{-1}a_i de_i,$$

where $f'(x)^{-1} \notin M$, $(-f'(x))^{-1}a_i \in R$. This proves the lemma.

Lemma 2.3. *Let $U = \{u_1, \ldots, u_s\}$ be a set of generators of the maximal ideal M in R. Then*

(1) $RdR = RdE + RdM$;
(2) $RdM = RdU + MdR$;
(3) $RdR = RdE + RdU + MdR$.

Proof. We only need to prove (2), because (1) is already contained in Lemma 2.2, and (3) is a combination of (1) and (2). Since $M = RU$, $RdM \subset RdU + MdR$. *But* $MdR \subseteq RdM$ (as $mda = d(ma) - adm$ for any $m \in M$ and $a \in R$). Thus $RdM = RdU + MdR$. Q.E.D.

Lemma 2.4. *Put $U \cup E = N$. If RdR/RdN is a finite R-module, then $RdR = RdN$.*

Proof. According to Lemma 2.3, (3), we have $RdR = RdN + MdR$, which, together with the assumption on RdR/RdN, implies that $RdR = RdN$ by Nakayama's lemma. Q.E.D.

Theorem 2.5. *Let R be a locality of K/F, E a lift of a L/k-differential basis $E^* < F$, and U a set of generators of M in R. Then $RdR = RdN$ (where $N = U \cup E$).*

Proof. According to Lemma 2.4, we only need to show that RdR/RdN is a finite R-module. Suppose $R = F[x_1, \ldots, x_s]_P$ and let B' be a F/k-differential basis such that $B' \sim E$. Denote $B' \cup \{x_1, \ldots, x_s\} = Q$. Then $RdR = RdQ$. Since $Q \sim B' \sim E \sim N$, $T = Q - (Q \cap N)$ is a finite set. From

$$RdR = RdQ = R(dT \cup d(Q \cap N)) = R(dT \cup dN)$$

it follows that RdR/RdN is a finite R-module. Q.E.D.

Our main theorem of this section is the following

THEOREM 2.6. *Let R be a regular locality of K/F, and k a subfield of F such that $[KF]_k = [LF]_k$. Then R has a set of normal uniformizing coordinates B_R over k with $B_R \sim B < F$. In fact, if U is a minimal basis of M in R, E a lift of a L/k-differential basis E^* such that $E \sim E^* \sim B$, then $N = U \cup E$ is such a set of uniformizing coordinates of R over k.*

Proof. By definition it is easy to see that a subset N of R is a set of uniformizing coordinates of R over k if and only if the following two conditions are satisfied:

(1) $RdR = RdN$;
(2) N is K/k-differentially independent.

Now (1) is true by Theorem 2.5, and (2) follows from Lemma 1.4 (e) since, for any F/k-differential basis B' with $B' \sim E$,

$$\begin{aligned} |NB'| = |EB'| + |NE| &= [LF]_k + \text{trans.deg } L/F + \text{krull dim } R \\ &= [KF]_k + \text{trans.deg } L/F + \text{krull dim } R \\ &= [KF]_k + \text{trans.deg } K/F. \end{aligned}$$

(Note $|NE| = |U| =$ krull dim R, since R is regular.) Q.E.D.

COROLLARY 2.7. *If k is a perfect subfield of F (e.g. the prime field of F), then any regular locality R of K/F has a set of normal uniformizing coordinates B_R over k with $B_R \sim B < F$.*

Proof. Since k is perfect, $[KF]_k = [LF]_k = 0$ by Lemma 1.4, (f). Thus we can apply the theorem. Q.E.D.

THEOREM 2.8. *Let R be a regular locality of K/F. Then $[KF]_k \leq [LF]_k$ for any subfield k of F.*

Proof. We use the notations of Theorem 2.5 (assume that U is a minimal base of M). Since $RdR = RdN$, N contains a K/k-differential basis N' with $N' \sim B'$ (by Lemma 1.4). Then $|NB'| = |EB'| + |NE| = [LF]_k +$ trans.deg $L/F +$ krull dim $R = [LF]_k +$ trans.deg K/F, $|N'B'| = [KF]_k +$ trans.deg K/F, and $|NB'| \geq |N'B'|$. It follows that $[LF]_k \geq [KF]_k$. Q.E.D.

COROLLARY 2.9. *If R is a regular locality of K/F with $[LF]_k = 0$ (e.g. if L/F is a separably generated extension, see Corollary 1.6), then*

$[KF]_k = [LF]_k = 0$, *and* R *has a set of normal uniformizing coordinates* B_R *over* k *such that* $B_R \sim B < F$.

LEMMA 2.10. (a) *Let* x *be an element of* R, $G = \{g_j\}$ *a subset of* R, *and* $x \notin G$. *If* $dx = \Sigma\, a_j dg_j + w$, *where* $w \in MdR$, *and* $a_j \in R$, *then* $G \cup \{x\}$ *is not* R*-independent over any subfield of* F.

(b) *Let* G *be a subset of* M. *If the set of residue classes of the elements of* G *in* M/M^2 *is not* R/M*-independent, then* G *is not* R*-independent over any subfield of* F.

(c) *Let* $B = \{b_j\}$ *be a set of uniformizing coordinates of* R *over* k, $G = \{g_j\}$ *a finite subset of* R, *and* $dg_j = \Sigma\, a_{jt} db_j$. *Then* G *is* R*-independent over* k *if and only if* rank $\|a_{jt}\|$ mod $M = |G|$.

Proof. Part (a) follows from Nakayama's lemma, while (b) and (c) can be proved easily by using (a). Q.E.D.

LEMMA 2.11. *If* R *is a regular locality and* $[KF]_k = [LF]_k$, *then a subset of* M *is a part of a minimal basis of* M *if and only if it is* R*-independent over* k.

Proof. The "only part" follows from Lemma 2.10, (b). For the other direction apply Theorem 2.6. Q.E.D.

THEOREM 2.12. (*Zariski's mixed Jacobian criterion* [Z2]). *Let* $F[X] = F[X_1, \ldots, X_s]$ *be a polynomial ring*, N *and* P *two prime ideals of* $F[X]$ *with* $N \subseteq P$, *and* $R = F[X]_P/NF[X]_P$. *Let* k *be a subfield of* F *and* $[LF]_k = 0$ (*where* $L = R/M$ *and* $K = QR$). *Let* $U = \{u_t\}$ *be a finite set of generators of* N, $B' = \{b_j'\}$ *a* F/k*-differential basis, and*

$$(*) \qquad du_t = \Sigma\, a_{tj} db_j' + \Sigma\, c_{tz} dX_z.$$

Let I *be the coefficient matrix of the right side of* (*) (*with respect to* $\{db_j'\} \cup \{dX_z\}$). *Then*

(1) R *is regular if and only if the rank of* I mod P *equals* $v =$ *height* N;

(2) *Let* $G = \{b_j' \in B' \mid a_{tj} \neq 0$ *for some* $t\}$ *and* $G' = G \cup \{x_1, \ldots, x_s\}$ *where* $k[x] = k[X]/N$. *Then* G *is a finite* F/k*-differentially independent subset of* F. *If* R *is regular, we can find a subset* G_1' *of* G' *with* v *elements such that* $(B' - G) \cup (G' - G_1')$ *is a set of uniformizing coordinates of* R *over* k.

Proof. (1) We may assume that U is a minimal basis of the maximal ideal of $F[X]_N$ (because the rank will be the same). We contend that the following assertions are equivalent:

(a) R is regular;
(b) U is a part of a minimal base of $PF[X]_p$;
(c) U is $F[X]_P$-independent;
(d) the rank of I mod P (or mod $PF[X]_P$) equals $|U| = \nu$.

The equivalence of (a) and (b) is well known. That of (b) and (c) follows from Lemma 2.11, as we have $[F(X)F]_k = [LF]_k = 0$. Finally the equivalence of (c) and (d) is the content of Lemma 2.10, (c), since $\{X\} \cup B'$ is a set of uniformizing coordinates of $F[X]_p$.

(2) If R is regular, then, according to part (1), the rank of I mod P equals ν = height N; we can choose a subset G_1' of G' with ν elements such that, if $H = (B' - G) \cup (G' - G_1')$, then $RdR = RdH$, and $|HB'|$ = trans.deg K/F by direct calculation. Since $[LF]_k = 0$, we have $[KF]_k = 0$ by Corollary 2.9. Applying Lemma 1.4, (e) we see that H is a K/k-differential basis. This shows that H is a set of uniformizing coordinates of R over k. Q.E.D.

Finally we introduce the concept of a "reference set," which is needed in the next section (see the proof of Lemma 3.1 below).

Definition 2.13. Let C be a set of regular localities of K/F. A subset G of F is called a reference set of C over a subfield k of F if the following conditions are satisfied:

(a) G is F/k-differentially independent;
(b) For any $R \in C$ there exists a subset $T \subset R$ such that, whenever B' is a F/k-differential basis containing G, then $(B' - G) \cup T$ is a set of uniformizing coordinates of R.

(Note that if G is finite, then T is also finite by Lemma 1.4.)

LEMMA 2.14. (a) *Let $C(A)$ be the set of regular localities of an affine domain $A = F[x_1, \ldots, x_s]$, and k a perfect subfield of F. Then $C(A)$ has a finite reference set over k.*

(b) *If $A_1, \ldots, A_m$ are affine domains of K/F and C' is the union of $C(A_1), \ldots, C(A_m)$ (as in* (a)*), then C' has a finite reference set over k.*

Proof. (a) has already been proved (see part (2) of the above theorem).

(b) Let G be a finite F/k-differentially independent subset of F containing, for each A_g, a finite reference set $G(A_g)$ of $C(A_g)$. Then it is easy to see that G is a finite reference set of C' over k. Q.E.D.

3. Generalized Jacobian of differential bases. Starting from this section, until the end of Section 5, we shall only consider absolute differentials, i.e. the differentials over the prime field. Therefore, in the following, when we speak of differentials of a field, we tacitly assume that they are taken over the prime field. We shall fix an algebraic function field, denoted by K/k (instead of K/F as we did in the last section), of dimension $n > 0$. By a differential basis of K/k we mean a differential basis B of K (over the prime field of k) with $B < k$. We denote the set of differential bases of K/k by $S(K/k)$, or simply by S. Let H (resp. H^1) be the set of regular localities (resp. regular localities of krull dim 1) of K/k. Henceforth, all differential bases of K (or any set of uniformizing coordinates for regular localities) and all regular localities under our investigation will be taken from the sets S and H respectively. We emphasize the fact that, since the prime fields are perfect, all the results of the previous sections are applicable here. For our purpose, the most important one is that *any regular locality of H has a set of uniformizing coordinates in S* (see Corollary 2.7).

Let B_1 and B_2 be two differential bases of K/k and suppose that $B_1 \sim B_2$. Up to a $\pm$ sign, we can define the determinant of the transformation matrix, from the linear basis dB_1 to dB_2, since there are only finite elements different from 1 (resp. from 0) on the diagonal (resp. away from the diagonal) of the matrix. Passing to the residue group K^*/k^* (K^* and k^* are the multiplicative groups of K and k respectively), we obtain a unique class, denoted by (B_2, B_1).

LEMMA 3.1. *Let V be a normal projective model of K/k and H_V^1 the set of regular local rings of* (regular) *points of V with codim* 1. *Then there exists a unique map*

$$J_V: S \times S \to K^*/k^*$$

satisfying the following conditions:

(1) $J_V(B_1, B_2).J_V(B_2, B_3) = J_V(B_1, B_3)$, *for any B_1, B_2, B_3 in S.*

(2) *If $B_1 \sim B_2$, then $J_V(B_1, B_2) = (B_1, B_2)$.*

(3) *Let B_R be a set of uniformizing coordinates of a regular locality R*

in H_V^1, *and* B_1 *a differential basis of* K/k *contained in* R. *Then* $J_V(B_1, B_R) = x^*$, *where* x^* *is the residue class of an element* $x \in R$ *in* K^*/k^*.

((1) and (2) imply that x is an invertible element of R if B_1 is also a set of uniformizing coordinates of R).

Proof. First we prove the uniqueness of J_V. Let J'_V be another map satisfying the same conditions. For any $B_1, B_2 \in S$ we shall prove that $J_V(B_1, B_2) = J'_V(B_1, B_2)$. Let $R \in H_V^1$ and let B_{1R}, B_{2R} be two sets of uniformizing coordinates of R such that $B_{1R} \sim B_1, B_{2R} \sim B_2$. Then by (2) we have

$$J_V(B_1, B_{1R}) = J'_V(B_1, B_{1R}) = (B_1, B_{1R}), \text{ and}$$

$$J_V(B_{2R}, B_2) = J'_V(B_{2R}, B_2) = (B_{2R}, B_2).$$

Thus

$$\begin{aligned} J_V(B_1, B_2)/J'_V(B_1, B_2) &= J_V(B_1, B_{1R})J_V(B_{1R}, B_{2R})J_V(B_{2R}, B_2)/ \\ &\quad J'_V(B_1, B_{1R})J'_V(B_{1R}, B_{2R})J'_V(B_{2R}, B_2) \\ &= J_V(B_{1R}, B_{2R})/J'_V(B_{1R}, B_{2R}) = x_1^*/x_2^*, \end{aligned}$$

where x_1, x_2 are units in R (by condition (3)). Since this is true for any regular locality R of K/k, we see that $x_1/x_2 \in \cap R = k$. Thus $x_1^*/x_2^* = 1^* \in K^*/k^*$ which shows that $J_V(B_1, B_2) = J'_V(B_1, B_2)$.

To define J_V we note that if we put $J_V(B_1, B_2) = (B_1, B_2)$ for any $B_1 \sim B_2$, then the conditions (1)-(3) are satisfied for any $B_1 \sim B_2 \sim B_3$. Since V is covered by finite affine open sets over k, we have seen in Section 2 that H_V^1 has a finite reference set $N \subseteq k$. Let B_1, B_2 be two differential bases of K/k, and Z_1, Z_2 two differential bases of k, such that $B_1 \sim Z_1, B_2 \sim Z_2$ and $N \subseteq Z_1, Z_2$. For any $R \in H_V^1$ we can find, according to the definition of a reference set, a finite subset G_R of K, such that $Q_{1R} = (Z_1 - N) \cup G_R$ and $Q_{2R} = (Z_2 - N) \cup G_R$ are two sets of uniformizing coordinates of R (G_R

depends on R, but not on Z, for a fixed N). Also we have $B_1 \sim Q_{1R}$ and $B_2 \sim Q_{2R}$. Define

$$J_V(B_1, B_2) = (B_1, Q_{1R})(Q_{2R}, B_2).$$

We have to show that $J_V(B_1, B_2)$ only depends on B_1 and B_2 (then we may consider J_V as a map from $S \times S$ to K^*/k^*). If N', R', $Q'_{1R'}$ and $Q'_{2R'}$ are other similar choices for N, R, Q_{1R} and Q_{2R}, with the corresponding $J'_V(B_1, B_2)$, then, for any $R'' \in H_V^1$, $Q_{1R''}$ (with respect to N), and $Q'_{2R''}$ (with respect to N') we have $(Q_{1R''}, Q_{1R}) = (Q_{2R''}, Q_{2R})$ and $(Q'_{1R'}, Q'_{1R''}) = (Q'_{2R'}, Q'_{2R''})$ by direct calculations. Then

$$\begin{aligned} &J_V(B_1, B_2)/J'_V(B_1, B_2) \\ &\quad = (B_1, Q'_{1R'})(Q'_{1R'}, Q_{1R})(Q_{2R}, Q'_{2R'})(Q'_{2R'}, B_2)/ \\ &\qquad (B_1, Q'_{1R'})(Q'_{2R'}, B_2) \\ &\quad = (Q'_{1R'}, Q_{1R})(Q_{2R}, Q'_{2R'}) = (Q'_{1R'}, Q_{1R})/(Q'_{2R'}, Q_{2R}) \\ &\quad = (Q'_{1R'}, Q'_{1R''})(Q'_{1R''}, Q_{1R''})(Q_{1R''}, Q_{1R})/ \\ &\qquad (Q'_{2R'}, Q'_{2R''})(Q'_{2R''}, Q_{2R''})(Q_{2R''}, Q_{2R}) \\ &\quad = (Q'_{1R''}, Q_{1R''})/(Q'_{2R''}, Q_{2R''}) = x_1^*/x_2^*, \end{aligned}$$

where x_1, x_2 are units in R''. Hence

$$J_V(B_1, B_2)/J'_V(B_1, B_2) = s^*,$$

s being a unit of $R'' \in H_V^1$. It follows again that $s \in \cap R'' = k$. Therefore $J_V = J'_V$.

It is easily seen that J_V, viewed as a map from $S \times S$ to K^*/k^*, satisfies the three conditions. This finishes the proof of Lemma 3.1.

The map J_V defined above is in fact independent of the choice of V. More specifically, we have

THEOREM 3.2. *For any algebraic function field K/k there exists a unique map*

$$J: S \times S \to K^*/k^*$$

satisfying the conditions (1), (2) *of Lemma* 3.1 *and* (3) *for any regular locality of K/k.*

Proof. The uniqueness of J is obvious because, if such a map exists, it must coincide with the map J_V defined in Lemma 3.1 for any normal projective model V of K/k.

Next we point out that if J is a map having the properties (1), (2), and (3) for any $R \in H^1$, then J already satisfies (3) for any regular localities of K/k. For if R is a regular locality of K/k, with B_R a set of uniformizing coordinates of R, and if ${H_R}^1$ is the set of quotient rings of R with respect to the prime ideals of R of height 1, then B_R is also a set of uniformizing coordinates for any R_P in ${H_R}^1$ (we have $d(a/b) = (bda - adb)/b^2 \in R_P dB$ for any $a, b \in R$ with $b \notin P$; thus $R_P dR_P = R_P dB$). Therefore $J(B_1, B_R) = x^*$, where $x \in \bigcap R_P = R$.

Finally we prove that the map J_V defined in Lemma 3.1 for any normal projective model V of K/k is in fact independent of the choice of V. Let V' be another normal projective model of K/k, and V'' the normalization of the graph of the birational map from V to V' (see [Z1]). Then ${H_V}^1$, ${H_{V'}}^1 \subseteq {H_{V''}}^1$. From the uniqueness of J_V and $J_{V'}$ we find that $J_{V''} = J_V$, and $J_{V''} = J_{V'}$. Hence $J_V = J_{V'}$. Denote this unique map by J. Since any regular locality R in H^1 is the local ring of a regular point of codim 1 of a normal projective model of K/k, we conclude that the map J satisfies the condition (3) for any R in H^1, hence, by the remark in the last paragraph, for any R in H. Since the conditions (1) and (2) of the theorem are the same as those given in Lemma 3.1, we see that J is the map we are looking for. Q.E.D.

The unique map J defined above is called the "Generalized Jacobian" of the differential bases of K/k.

4. Canonical ring of algebraic function fields. Now we come to the global part of the theory. This section is devoted to constructing the canonical ring for an arbitrary algebraic function field K/k. We shall use the same notations as in Section 3.

Definition 4.1. A universal algebra of K/k is a graded k-algebra $A = \bigoplus_{i \geq 0} A_i$ which is at the same time a polynomial ring $A = \bigoplus_{i \geq 0} A_i = \bigoplus_{i \geq 0} KX^i$ in one variable $X \in A_1$ over K. A_i is the i-th homogeneous component of A. An element of A_i is called a homogenous element of degree i.

We apply the "generalized Jacobian" map $J: S \times S \to K^*/k^*$, established in the preceding section, to define a canonical algebra for K/k.

Definition 4.2. A canonical algebra of K/k is a pair (A, τ) consisting of a universal algebra A of K/k, together with a map $\tau: S \to A_1$, such that, for any $B_1, B_2 \in S$, we have $\tau(B_2) = a\tau(B_1) \neq 0$, where $a \in K$ and $a^* = J(B_2, B_1)$.

Suppose a universal algebra A of K/k is given. To construct such a map τ, take any $B \in S$, $X \in A_1$ and assign $\tau(B) = X$. Then τ can be extended (noncanonically) to a map from S to A_1, such that, for any $B_1 \in S$, $\tau(B_1) = a\tau(B)$ with $a^* = J(B_1, B)$. The same formula also holds for any B_1, $B_2 \in S$, in view of Theorem 3.2 (1). Therefore the map τ, together with A, forms a canonical algebra of K/k.

Let (A, τ) be a canonical algebra of K/k, R a regular locality of K/k. According to Corollary 2.7, R has a set of uniformizing coordinates B_R in S. The image $\tau(B_R)$ generates a polynomial ring $\tau(R) = \bigoplus_{i \geq 0} R(\tau(B_R))^i$ over R in A. If B_R' is another set of uniformizing coordinates of R in S, we have $\tau(B_R') = a\tau(B_R)$, with $a^* = J(B_R', B_R)$. By Theorem 3.2 (3), a is a unit of R. Thus $\tau(B_R)$ and $\tau(B_R')$ generate the same polynomial ring $\tau(R)$ over R. This shows that $\tau(R)$ is uniquely determined by R.

Definition 4.3. The canonical ring of K/k (associated with a canonical algebra (A, τ)) is the intersection

$$C(K/k) = \bigcap \tau(R),$$

where R runs through all regular localities of K/k. (In the following we shall simply write C for $C(K/k)$ if no confusion will thereby arise)

Since $k \subseteq \tau(R)$ for each R, we have $k \subseteq C$, which implies that C is not empty, and we may regard C as a k-algebra. Furthermore, since each $\tau(R)$ is a graded k-algebra, C has a natural graded k-algebra structure: $C = \bigoplus_{i \geq 0} C_i$, with $C_i \subseteq A_i$ as the i-th component of C.

We show that the canonical ring C is uniquely determined by K/k, up to a k-graded algebra isomorphism. Take another canonical algebra $(A',$ $\tau')$ with the corresponding canonical ring $C' = \bigoplus_{i \geq 0} C'_i$, and fix an arbitrary differential basis B of K/k. Since both A and A' are polynomial rings in one variable over the field K, there is a unique K-algebra isomorphism $f: A \to A'$ such that $f(\tau(B)) = \tau'(B)$. If B_R is a set of uniformizing coordinates for a regular locality R of K/k, then $\tau(B_R) = a\tau(B)$, $\tau'(B_R) = a'\tau'(B)$ with $a/a' \in k$ (as $a^* = a'^*$ in K^*/k^*). Therefore $\tau(R) = \oplus_{i\geq 0} R(\tau(B_R))^i = \oplus_{i\geq 0} R(a\tau(B))^i$ is mapped onto $\tau'(R) = \oplus_{i\geq 0} R(\tau'(B_R))^i = \oplus_{i\geq 0} R(a'\tau'(B))^i$ by f. Since $C = \cap\tau(R)$ and $C' = \cap\tau'(R)$, we see that $f|_C: C \to C'$ is a k-graded algebra isomorphism.

For the remainder of this section we study some elementary properties of C. We need a relative version of C. Let H be the set of regular localities of K/k (as in the last section), and H' a subset of H. Define $C(H') = \cap\tau(R)$, as R runs through all $R \in H'$. In general, C is contained in $C(H')$ as a k-graded subalgebra. If H^1 is the set of regular localities of krull dim 1 of K/k, then we have

LEMMA 4.4. $C = C(H^1)$.

Proof. Since any $R \in H$ is integral closed, we have $R = \cap R_P$, where the intersection is taken over all the prime ideals of height 1 of R. Let B be a set of uniformizing coordinates of R. Then B is also a set of uniformizing coordinates for any R_P. Thus

$$\tau(R) = \bigoplus_{i\geq 0} R(\tau(B))^i = \cap \bigoplus_{i\geq 0} R_P(\tau(B))^i = \cap\tau(R_P).$$

It follows that

$$C = C(H) = \cap\tau(R) = \cap(\cap\tau(R_P)) = C(H^1). \qquad \text{Q.E.D.}$$

Let $w_i \neq 0$ be a homogeneous element of A_i. For a normalized discrete valution v (of rank 1) of K/k, we associate an integer $v(w_i)$ in the following way:

Denote $R_v = \{z \in K | v(z) \geq 0\}$. Then R_v is a regular locality of K/k of krull dim 1. Let B_R be a set of uniformizing coordinates of R_v. Then $w_i = a(\tau(B_R))^i$ for some $a \in K$. Define $v(w_i) = v(a)$. Clearly $v(w_i)$ is independent of the choice of B_R.

The integer $v(w_i)$ may be $\neq 0$ for infinitely many v. But if we only treat the discrete valutations v_W with respect to the subvarieties W^{n-1} of a normal complete model V^n of K/k, here $n = \text{trans.deg}\, K/k$, then $v_W(w_i) \neq 0$ for only a finite number of v_W. Define $(w_i)_V = \Sigma\, v_W(w_i)W$. Then $(w_i)_V$ is a

Weil divisor of V. If $w_i' \neq 0$ is another element of A_i, we have $w_i' = bw_i$ for some $b \in K$. It follows that $(w_i')_V = (b)_V + (w_i)_V$. Thus $(w_i')_V$ is linearly equivalent to $(w_i)_V$. The unique divisor class determined by $(w_i)_V$ is called the i-th canonical class of V.

Now assume $C_i \neq 0$ for some $i > 0$ and fix one of its element $w_i \neq 0$. The map $f: C_i \to K$ given by $f(w_i') = w_i'/w_i$ for any $w_i' \in C_i$ is an embedding of C_i into the k-vector space $M = \{z \in K | (z)_V + (w_i)_V \geq 0\}$. Since $\dim_k M < +\infty$ as is well known, we have

LEMMA 4.5. $\dim_k C_i < +\infty$ *for all* $i \geq 0$.

Definition 4.6. $g_i = \dim_k C_i$ is the i-th pluri-genus of K/k.

THEOREM 4.7. *If V is a nonsingular complete model of K/k, and H_V^1 the set of prime divisors of V (i.e. the set of regular localities of the points of V with* codim 1), *then* $C(H_V^1) = C$.

Proof. Any $R \in H^1$ is a discrete valuation ring with the discrete valuation v_R of K. Since V is a complete model, v_R has a center P in V, and the lcoal ring R' of P is regular, with $R' \subseteq R$. Let B_R and B_R' be two sets of uniformizing coordinates of R and R' respectively. Then $\tau(B_R') = a\tau(B_R)$ with $a \in R$ by Theorem 3.2, (3), and $R'(\tau(B_R'))^i = R'(a\tau(B_R))^i = R'a^i(\tau(B_R))^i \subseteq R(\tau(B_R))^i$. On the other hand, since $R'(\tau(B_R'))^i = \cap R_P'(\tau(B_R'))^i$, where P runs through the prime ideals of height 1 of R', we have $C(H_V^1) \subseteq C(H^1) = C(H)$. Combining this with the obvious relation $C(H_V^1) \supseteq C$, we find that $C(H_V) = C$. Q.E.D.

THEOREM 4.8. *$C(H')$ is integrally closed for any subset H' of H. Particularly, the canonical ring of K/k is integrally closed.*

Proof. By definition $C(H') = \cap\, \tau(R)$ with $R \in H'$, and each $\tau(R)$ is a polynomial ring in one variable over R, hence an integrally closed domain (because R is one; see [M], p. 116). It follows that $C(H) = \cap\, \tau(R)$ is an integrally closed domain.

Let $H' = H$, we obtain the last assertion. Q.E.D.

5. Kodaira dimension of algebraic function fields. In this section K/k will be a fixed algebraic function field of dimension $n > 0$, (A, τ) a canonical algebra for K/k, and C the canonical ring of K/k associated with (A, τ).

Definition 5.1. The *canonical field* of K/k is the subfield $Z(K/k) = QC \cap K$ of K, here QC is the quotient field of C. We call the integer

$\kappa(K/k) = \text{trans.deg}\ QC/k - 1$ the *Kodaira dimension* of K/k. If V is a normal complete model of K/k, we define the Kodaira dimension of V by

$$\kappa(V) = \text{trans.deg}\ QC(H_V^1)/k - 1.$$

As C is uniquely determined by K/k (up to a graded k-algebra isomorphism), the canonical field and the Kodaira dimension of K/k are all invariants of K/k.

LEMMA 5.2. *If* $\kappa(K/k) \neq -1$, *then* $\text{trans.deg}\ QC/Z(K/k) = 1$, *and* $\kappa(K/k) = \text{trans.deg}\ Z(K/k)/k$.

Proof. The second assertion follows immediately from the first one. Clearly, our assumption implies that $C_i \neq 0$ for some $i > 0$, because otherwise $QC = C_0 = k$. Fix an element $x \neq 0$ of C_i. Let $y \neq 0$ be an element of any $C_j \neq 0$ with $j > 0$. Then there exists an element $a \in Z(K/k)$ such that $y^i = ax^j$. Thus y is algebraic over $Z(K/k)(x)$. Since QC is the quotien field of C, we conclude that $\text{trans.deg}\ QC/Z(K/k) = 1$. Q.E.D.

THEOREM 5.3. *The subfields* $Q(C(H'))$ *and* $Z'(K/k) = Q(C(H')) \cap K$ *are algebraically closed in* QA *for any* $H' \subseteq H$. *The canonical field* $Z(K/k)$ *of* K/k *is algebraically closed in* K.

Proof. Suppose that $z \in QA$ is algebraic over $Q(C(H'))$. There exists an element a of $C(H')$ such that az is integral over $C(H')$. As $C(H') = \cap \tau(R)$, where R runs through H', the element az is integral over each $\tau(R)$. Since $\tau(R)$ is integrally closed and $Q(\tau(R)) = QA$, we see that $az \in \tau(R)$ for all $R \in H'$. Thus $az \in C(H')$ and $z = az/a \in QC(H')$. Therefore $Q(C(H'))$ is algebraically closed in QA. Since K is algebraically closed in QA, the field $Z'(K/k)$, as the intersection of $Q(C(H'))$ and K, is algebraically closed in QA. Let $H' = H$ we conclude that the canonical field $Z(K/k)$ of K/k is algebraically closed in QA, therefore also in K. Q.E.D.

LEMMA 5.4. $\kappa(K/k) = -1$ *if and only if* $C = k$; $\kappa(K/k) = 0$ *if and only if* $\dim_k C_i \leq 1$ *for any* $i > 0$, *and the equality holds for at least one* $i > 0$.

Proof. The first part is obvious. To prove the second assertion, assume that $\dim_k C_i > 1$ for some $i > 0$ and let $z_1, z_2 \in C_i$ be two linearly independent elements over k. Then $z_1/z_2 \in Z(K/k)$ and $z_1/z_2 \notin k$. Since k is algebraically closed in K, z_1/z_2 is transcendental over k. But trans.deg $Z(K/k)/k > 0$ implies that $\kappa(K/k) > 0$ by Lemma 5.2. Q.E.D.

Let F be a subfield of K containing k, and suppose that K/F is also an algebraic function field. We have $S(K/F) \supseteq S(K/k)$. It is easy to see that, in view of Corollary 2.7, any regular locality of K/F has a set of uniformizing coordinates in $S(K/k)$. Take a canonical algebra (A, τ') of K/F such that $\tau'(B) = \tau(B)$ for any $B \in S(K/k)$. The canonical ring $C(K/F)$ associated with (A, τ') is then independent of the choice of the map τ'. Since $H^1(K/F) \subseteq H^1(K/k)$, we see that $C(K/F)$ contains $C(K/k)$ as a graded k-algebra. We shall call (A, τ') an induced canonical algebra of K/F (with respect to the canonical algebra (A, τ) of K/k).

THEOREM 5.5. (*Fibering Theorem*) (*cf.* [I2], *p.* 310). *Let K/k be an algebraic function field of $0 \leq \kappa(K/k) < n$. Let $Z(K/k)$ be the canonical field of K/k. If K/k has a normal complete model V such that $QC(H_V{}^1) = QC$ (or equivalently, if $\kappa(V) = \kappa(K/k)$), then the Kodaira dimension $\kappa(K/Z(K/k))$ of $K/Z(K/k)$ is zero.*

Proof. Write Z for $Z(K/k)$ and G for $H_V{}^1$. From the above argument we obtain the canonical ring $C(K/Z)$ of K/Z associated with an induced canonical algebra with respect to (A, τ), and we have $C(K/Z) \supseteq C(K/k)$. Since $\kappa(K/k) \geq 0$, we have $C(K/k) \not\subset K$. Thus $C(K/Z) \not\subset K$ which shows that $\kappa(K/Z) \geq 0$. Let G' be the set of regular localities in G containing the field Z. Then $C(G') \supseteq C(K/Z)$. We propose to prove that $\dim_Z C_i(G') \leq 1$ for those i with $C_i(G) \neq 0$. This will imply that $\dim_Z C_i(K/Z) \leq 1$ for all i, which means $\kappa(K/Z) = 0$ by Lemma 5.4.

Fix an element $z \neq 0$ of $C_i(G)$. We shall show that, for any nonzero element $x \in C_i(G')$, the quotient x/z is in Z, which will finish the proof. We only need to consider those $x \notin C_i(G)$. For if $x \in C_i(G)$, then x/z is already contained in $QC(G) \cap K = Z$ by the choice of V.

Thus let $x \notin C_i(G)$. This implies that there exists a finite subset $\{R_j\}$ of G such that $v_{R_j}(x) = -s_j < 0$. Clearly $R_j \notin G'$ as $x \in C_i(G')$. Thus $Z \not\subset R_j$. Since each R_j is a discrete valuation ring, we can find a nonunit a_j/b_j of R_j in Z for each R_j (here a_j, b_j are homogenous elements of the same degree in $C(G)$). Then $v_{R_j}(a_j) > 0$. Consider the product $x' = x \prod a_j^{s_j}$. Since $v_R(x') \geq 0$ for any $R \in G$, we see that $x' \in C(G)$. But the product $z' = z \prod a_j^{s_j}$ is a homogenous element of $C(G)$ with the same degree as x'. Therefore the quotient $x'/z' = x/z \in Z$. The proof is complete.

THEOREM 5.6. *Let F be a subfield of K containing k and suppose that K/F is an algebraic function field. Then*

(1) $Z(K/F) \supseteq Z(K/k)$;
(2) *if* $\kappa(K/k) \geq 0$, *then* $\kappa(K/F) \geq 0$, *and* $\kappa(K/k) \leq \kappa(K/F) + \dim F/k$ (*cf.* [I2], *p.* 310. "*Easy Addition Theorem*");
(3) *if* $\kappa(K/F) = 0$, *then* $F \supseteq Z(K/k)$.

Proof. (1) Let $C(K/F)$ be the canonical ring of K/F associated with a canonical algebra induced from (A, τ). Then $C(K/F) \supseteq C$. It follows that

$$C(K/F) \cap K \supseteq C \cap K$$

which proves (1).

(2) We have $C(K/F) \supseteq C$; if $\kappa(K/k) \geq 0$, then C is not contained in K. It follows that $C(K/F)$ is not contained in K. Therefore $\kappa(K/F) \geq 0$. For the inequality we have

$$\begin{aligned} \kappa(K/k) &= \text{trans.deg } Z(K/k)/k \\ &\leq \text{trans.deg } Z(K/F)/k \\ &= \text{trans.deg } Z(K/F)/F + \text{trans.deg } F/k \\ &= \kappa(K/F) + \dim F/k. \end{aligned}$$

(3) If $\kappa(K/F) = 0$, we have $Z(K/F) = F$ by Lemma 5.2, and the assertion follows immediately from this and (1). Q.E.D.

THEOREM 5.7. (*cf.* [I2], *p.* 311. "*Uniqueness Theorem*"). *Let K/k be an algebraic function field of $0 \leq \kappa(K/k) < n$. Let F be a subfield of K containing k such that K/F is an algebraic function field and $\kappa(K/F) = 0$.*

(1) $F = Z(K/k)$ *if and only if* $\dim F/k = \kappa(K/k)$.

(2) *Suppose K/k has a normal complete model V such that $\kappa(V) = \kappa(K/k)$. Then the canonical subfield $Z(K/k)$ of K/k is characterized by the following two properties*:

(a) $\kappa(K/Z(K/k)) = 0$;
(b) $Z(K/k) \subseteq F$ *for any subfield F of K given above.*

Proof. We have $\kappa(K/k) = \dim Z(K/k)/k$, and $F \supset Z(K/k)$ by Theorem 5.6, (3). If $\dim F/k = \kappa(K/k)$, then $\dim F/k = \dim Z(K/k)/k$. Thus F is algebraic over $Z(K/k)$. But $Z(K/k)$ is algebraically closed in K, by Theorem 5.3. It follows that $F = Z(K/k)$. The other direction is obvious.

The assertion (2) follows easily from Theorem 5.5 and Theorem 5.6, (3) in a similar way. Q.E.D.

An algebraic function field K/k is said to be of elliptic (resp. parabolic, resp. hyperbolic) type if $\kappa(K/k) = -1$ (resp. $\kappa(K/k) = 0$, resp. $\kappa(K/k) = n$); K/k is of fiber type if $0 < \kappa(K/k) < n$.

Definition 5.8. The canonical series of K/k is the chain of subfields of K

$$K = Z_0 \supsetneq Z_1 \supsetneq \cdots \supsetneq Z_r = k$$

in which $Z_i = Z(Z_{i-1}/k)$ for $0 < i < r$, and Z_{r-1}/k is not of fiber type. (It is easy to see that Z_{i-1}/k is an algebraic function field for $0 < i < r$.)

Applying Theorem 5.7 repeatedly we obtain the following "towering theorem" for algebraic function fields:

THEOREM 5.9. *Suppose for any subfield F of K with $F \supseteq k$ and* dim *$F/k > 0$, there exists a normal complete model V of F/k such that $\kappa(V) = \kappa(F/k)$. Then K/k can be uniquely factored into a series of extensions*:

$$K = F_0 \supsetneq F_1 \supsetneq \cdots \supsetneq F_r = k$$

such that

(a) *each F_i is algebraically closed in K;*
(b) *$\kappa(F_{i-1}/F_i) = 0$ for $0 < i < r$;*
(c) *$\kappa(F_{i-1}/k) = \dim F_i/k$ for $0 < i < r$;*
(d) *F_{r-1}/k is not of fiber type.*

The chain $(F_0 \supsetneq F_1 \supsetneq \cdots \supsetneq F_r)$ is the canonical series of K/k.

THEOREM 5.10. *Let F be a subfield of K containing k and suppose that K/F is a separably algebraic extension. Then*

(1) *$\kappa(K/k) \geq \kappa(F/k)$;*

(2) *suppose that $\kappa(F/k) = n$; then $\kappa(K/k) = n$, and if $K \neq F$, there exists a positive integer $i > 0$ such that $p_i(K/k) > p_i(F/k)$ (therefore K/k*

and F/k are not isomorphic as abstract algebraic function fields over k; cf. [I2], *p.* 312. *Prop.* 10.10).

Proof. (1) Since K/F is a separably algebraic extension, we have $S(F/k) \subset S(K/k)$ according to Theorem 1.5, (1). Let $A'' = \bigoplus_{i \geq 0} F(\tau(B'))^i \subseteq A$, where B' is any differential basis of F/k, and let $\tau'' = \tau|_{S(F/k)\times S(F/k)}$. Then (A'', τ'') is a canonical algebra for F/k with the associated canonical ring $C(F/k)$. We shall prove that $C_i(F/k) \subseteq C_i(K/k)$. Suppose that w_i is an element of $C_i(F/k)$, and R' any regular locality of $H^1(K/k)$. Let $R = R' \cap F$. Then $R \in H^1(F/k)$. Let B be a set of uniformizing coordinates of R. Then we have

$$w_i = b(\tau''(B))^i$$

with $b \in R$ as $w_i \in C_i(F/k)$. Since $R \subseteq R'$, we also have $b \in R'$ and $B \subset R'$. It follows that

$$w_i = b(\tau''(B))^i \in \tau(R')$$

which shows that $C_i(F/k) \subset C_i(K/k)$. Thus $C(F/k)$ is a k-graded subalgebra of $C(K/k)$. Now we have

$$\begin{aligned} \kappa(K/k) &= \text{trans.deg } C(K/k) - 1 \\ &\geq \text{trans.deg } C(F/k) - 1 \\ &= \kappa(F/k). \end{aligned}$$

(2) If $\kappa(F/k) = n$, then $\kappa(K/k) = n$ by (1). Suppose that $p_i(K/k) = p_i(F/k)$ for all $i \geq 0$, i.e. $C_i(K/k)$ and $C_i(K/F)$ have the same dimensions over k. Since $C_i(F/k) \subset C_i(K/k)$, we see that $C_i(F/k) = C_i(K/k)$ for all $i \geq 0$. Therefore $C(F/k) = C(K/k)$. It follows that

$$\begin{aligned} K &= QC(K/k) \cap K \\ &= \{\text{the subfield of } QC(K/k) \text{ generated by the quotients of homogeneous elements of } C(K/k) \text{ of the same degree}\} \end{aligned}$$

$$= QC(F/k) \cap F$$

$$= F. \qquad \text{Q.E.D.}$$

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