Clone Theory and Algebraic Logic I

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Abstract

In this paper first-order logic with terms is interpreted in the framework of universal algebra using clone theory. Any first-order language determines a clone of terms and a predicate algebra of formulas over the clone. It is easy to translate the classical treatment of logic into our setting and prove the fundamental theorems of first-order theory algebraically.

Introduction

The theory of clones has been introduced in two previous papers [2] and [3]. In the present paper we are mainly concerned with the applications of clone theory to mathematical logic, as an extension of the last two sections of [3].

1 Clones

A *monoid* is an algebra $(G, \cdot, 1)$ where $1 \in G$ is called the *identity* and $\cdot : G \times G \to G$ is a binary operation such that for any $u, v, w \in G$ we have

$u \cdot (v \cdot w) = (u \cdot v) \cdot w.$

$1 \cdot u = u \cdot 1 = u.$

A *right act over a monoid* G is a pair (P, Θ) where P is a set and $\Theta : P \times G \to P$ is a multiplication such that for any $p \in P$ and $u, v \in G$ we have

$(pu)v = p(uv).$

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$p1 = p$.

Let $\mathcal{N}$ be the set of positive integers. If A is any set denote by $A^{\mathcal{N}}$ the set of infinite sequences $[a_1, a_2, ...]$ of elements of A.

A *clone* is a set A such that

(C1) $A^{\mathcal{N}}$ is a monoid with an identity $[x_1, x_2, ...]$.

(C2) A is a right act over $A^{\mathcal{N}}$.

(C3) $x_i[a_1, a_2, ...] = a_i$ for any $i > 0$.

Alternatively a clone can be defined as a set A containing a set $X = \{x_1, x_2, ...\}$ together with a multiplication $A \times A^{\mathcal{N}} \to A$ such that for any $a, a_1, a_2, ..., b_1, b_2, ... \in A$ we have

(D1) $(a[a_1, a_2, ...])[b_1, b_2, ...] = a[a_1[b_1, b_2, ...], a_2[b_1, b_2, ...], ...]$.

(D2) $a[x_1, x_2, ...] = a$.

(D3) $x_i[a_1, a_2, ...] = a_i$ for any $i > 0$.

2 Transformation Algebras

Let C be a concrete category over the category of sets (such as the of category sets, lattices or Boolean algebras, or any variety). We shall write compositions in C from left to right. If P is an object in C we write P^P for $hom(P, P)$. Then P^P is a monoid of transformations on P.

We define a *transformation algebra over* a clone A to be a pair (P, Θ) where P is an object in a concrete category C and $\Theta : P \times A^{\mathcal{N}} \to P$ is a multiplication such that the induced mapping $\Theta^* : A^{\mathcal{N}} \to P^P$ is a homomorphism of monoids. Algebraically this means that the following conditions are satisfied for any $p \in P$ and $a_1, a_2, ..., b_1, b_2, ... \in A$:

(T1) $(p[a_1, a_2, ...])[b_1, b_2, ...] = p[a_1[b_1, b_2, ...], a_2[b_1, b_2, ...], ...]$.

(T2) $p[x_1, x_2, ...] = p$.

(T3) The function $\phi_{[a_1,a_2,...]} : P \to P$ defined by $\phi(p) = p[a_1, a_2, ...]$ is an endomorphism on P.

If C is a finitary variety (of algebras) then (T3) has the following explicit form:

(T4) For any n-ary operation $f : P^n \to P$ on P we have $(f(p_1, ..., p_n))[a_1, a_2, ...] = f(p_1[a_1, a_2, ...], ..., p_n[a_1, a_2, ...])$.

Suppose P is a transformation algebra.

Let $p^+ = p[x_2, x_3, ...]$,

$p^- = p[x_1, x_1, x_2, x_3, ...]$.

$p^* = p[x_2, x_2, x_3, x_4, ...]$.

Then $(p^+)^- = p$ and $(p^-)^+ = p^*$

If $p \in P$ and $[a_1, a_2, ...] \in A^{\mathcal{N}}$, for any $n > 0$ we write $p[a_1, ..., a_n]$ as an abbreviation for $p[a_1, ..., a_{n-1}, a_n, a_n, a_n, a_n, ...]$.

We say P is *locally finite* if for any p there is $n > 0$ (called a *rank of* a) such that $p = p[x_1, ..., x_n]$. An element $p \in P$ is called *closed* (or *with a rank* 0) if $p[a_1, a_2, ...] = p$ for any $[a_1, a_2, ...] \in A^{\mathcal{N}}$.

A transformation algebra over A in the category of sets, lattices, Boolean algebras, ... is called a *transformation set, transformation lattice, transformation Boolean algebra*, ... over A. Note that a transformation set over A is just a right act over $A^{\mathcal{N}}$. Thus a clone A is a transformation set over itself.

3 Binding Operations

An *abstract binding operation* on a transformation algebra P over a clone A is a function $\forall : P \to P$ such that for any $p \in P$ and $[a_1, a_2, ...] \in A^{\mathcal{N}}$ we have

$$(\forall p)[a_1, a_2, ...] = \forall(p[x_1, a_1^+, a_2^+, ...]).$$

If $\forall$ is an abstract binding operation, for any positive integer i the conventional *i-th binding operation* $\forall x_i$ on P is defined by

$$\forall x_i.p = \forall(p[x_2, x_3, ..., x_{i-1}, x_i, x_1, x_{i+2}, ...]).$$

If $p \in P$ has a finite rank $n > 0$ then $\forall^n p$ is closed.

We have $\forall x_1.p = \forall(p[x_1, x_3, x_4, ...]) = (\forall p)^+$. So $\forall p = (\forall x_1.p)^-$.

4 Proposition and Boolean Algebras

A *proposition algebra* is an algebra $(P, \wedge, \neg)$ where $\wedge : P \times P \to P$ and $\neg : P \to P$ are functions.

If P is a proposition algebra for any $p, q \in P$ let

$p \vee q = \neg(\neg p \wedge \neg q)$.

$p \to q = \neg p \vee q$.

$p \leftrightarrow q = (p \to q) \wedge (q \to p)$.

$0 = \{p \wedge \neg p | p \in P\}$.

$1 = \{p \vee \neg p | p \in P\}$.

A *Boolean algebra* is a proposition algebra $(P, \wedge, \neg)$ satisfying the following conditions for any $p, q, r \in P$:

(B1) $p \wedge (q \wedge r) = (p \wedge q) \wedge r$.

(B2) $p \wedge q = q \wedge p$.

(B3) If $p \wedge \neg q = r \wedge \neg r$ then $p \wedge q = p$.

(B4) If $p \wedge q = p$ then $p \wedge \neg q = r \wedge \neg r$.

If P is a Boolean algebra then 0 and 1 are singletons and $(P, \vee, \wedge, 0, 1)$ is a complemented distributive lattice with the partial order on P defined by

$$p \leq q \Leftrightarrow p \wedge q = p.$$

for all elements p and q in P.

5 Predicate and Quantifier Algebras

Let A be a clone.

A *predicate (proposition) algebra* $(P, \forall)$ over A consists of a transformation proposition algebra P over A together with an abstract binding operation $\forall$ on P.

A *predicate (proposition) algebra with equality* $(P, \forall, e)$ over A consists of a transformation proposition algebra P over A, an abstract binding operation $\forall$ on P, and an element $e \in P$ of rank 2.

A *quantifier (Boolean) algebra* $(P, \forall)$ over A consists of a transformation Boolean algebra P over A and an abstract binding operation $\forall$ on P satisfying the following conditions for any $p, q \in P$:

(Q1) $\forall(p \wedge q) = \forall p \wedge \forall q$.

(Q2) $(\forall p)^+ = (\forall p)^+ \wedge p$.

(Q3) $\forall(p^+) = p$.

A *quantifier (Boolean) algebra with equality* $(P, \forall, e)$ over A consists of a quantifier algebra $(P, \forall)$ and an element $e \in P$ of rank 2 satisfying the following two conditions:

(E1) $e^* = 1$.

(E2) $e \wedge p = e \wedge p^*$.

The axioms (Q1), Q(2), (Q3) and (E1), (E2) are justified by the following observations:

1. Any abstract binding operation on a predicate algebra satisfying the axioms (Q1), Q(2) and (Q3) is unique if exists.

2. Any element in a quantifier algebra satisfying the axioms (E1) and (E2) is unique if exists.

3. There are plenty of concrete quantifier algebras (see Section 6).

We say a quantifier algebra $(P, \forall)$ is *nontrivial* if $0 \neq 1$.

We say a quantifier algebra $(P, \forall)$ is *simple* if $0 \neq 1$ and these are the only closed elements of P.

The class of predicate algebras (resp. quantifier algebras) over the same clone forms a finitary variety.

In the same way we obtain the varieties of predicate (resp. quantifier) Post algebras, Heyting algebras, frames, etc.

One can show that the variety of locally finite quantifier Boolean algebras over the initial clone X is equivalent to the variety of locally finite polyadic algebras of countably infinite degree (cf. [1]). Thus a quantifier algebra over an arbitrary clone may be viewed as a polyadic algebra with terms.

6 Interpretations

A *left algebra over a clone* A is a set M together with a multiplication $A \times M^{\mathcal{N}} \to M$ such that for any $a \in A$, $[a_1, a_2, ...] \in A^{\mathcal{N}}$ and $[m_1, m_2, ...] \in M^{\mathcal{N}}$ we have

(L1) $(a[a_1, a_2, ...])[m_1, m_2, ...] = a[a_1[m_1, m_2, ...], a_2[m_1, m_2, ...], ...]$.

(L2) $x_i[m_1, m_2, ...] = m_i$ for any $i > 0$.

If M is a left algebra over A let $P(M) = 2^{M^{\mathcal{N}}}$ be the set of functions from $M^{\mathcal{N}}$ to the Boolean algebra $2 = \{0, 1\}$ with two elements $0, 1$. Then $P(M)$ is a transformation Boolean algebra over A.

Define $\forall : P(M) \to P(M)$ such that for any $p \in M$ and $[m_1, m_2, ...] \in M^{\mathcal{N}}$ we have $(\forall p)[m_1, m_2, ...] = 1$ iff $p[m, m_1, m_2, ...] = 1$ for any $m \in M$. Then $\forall$ is an abstract binding operation on $P(M)$. Let $e \in P(M)$ be defined by $e[m_1, M_2, ...] = 1$ iff $m_1 = m_2$. Then $(P(M), \forall, e)$ is a quantifier algebra with equality over A, called the *function quantifier algebra determined by* M.

An *interpretation* for a predicate algebra P over a clone A is a pair (M, α) where M is a left algebra over A and α is a homomorphism of predicate algebras from P to $P(M)$. If P is a predicate algebra with equality e we also require that α preserves equalities.

Theorem 1 *(Godel's Completeness Theorem) Any nontrivial locally finite quantifier algebra (with or without equality) over a locally finite clone has an interpretation.*

7 Peano Algebras

Let $F_a = (\mathbf{0}, {}', +, \cdot)$ be a set of symbols. Let $X = \{x_1, x_2, ...\}$ be a set of variables. Let $F_a(X)$ be the smallest set such that $X \cup \{\mathbf{0}\} \subset F_a(X)$ and if $s, t \in F_a(X)$ then the expressions $s', s + t, s \cdot t$ are in $F_a(X)$.

Define a multiplication $F_a(X) \times F_a(X)^{\mathcal{N}} \to F_a(X)$ inductively for any $s, t, t_1, t_2, ... \in F_a(X)$:

$x_i[t_1, t_2, ...] = t_i$.

$\mathbf{0}[t_1, t_2, ...] = \mathbf{0}$.

$s'[t_1, t_2, ...] = (s[t_1, t_2, ...])'$.

$(s+t)[t_1, t_2, ...] = s[t_1, t_2, ...] + t[t_1, t_2, ...]$.

$(s \cdot t)[t_1, t_2, ...] = s[t_1, t_2, ...] \cdot t[t_1, t_2, ...]$.

Then $F_a(X)$ is a locally finite clone, called the *arithmetic clone*.

A *Peano algebra* is a quantifier algebra P over $F_a(X)$ generated by an equality $e \in P$ such that the following elements are equal to 1:

(S1) $\neg(e[\mathbf{0},\, x_1'])$.

(S2) $e[x_1', x_2'] \to e[x_1, x_2]$.

(S3) $e[x_1 + \mathbf{0},\, x_1]$.

(S4) $e[x_1 + x_2', (x_1 + x_2)']$.

(S5) $e[x_1.\mathbf{0},\, \mathbf{0}]$.

(S6) $e[x_1.(x_2)', (x_1.x_2) + x_1]$.

(S7) $(p[\mathbf{0}] \wedge (\forall(p[x_1] \to p[x_1']))) \to \forall p[x_1]$ for any $p \in P$.

Theorem 2 *(Godel-Rosser incompleteness theorem) There is a Peano algebra which is not simple.*

The proofs for Theorem 1 and Theorem 2 will be given in subsequent papers.

References

[1] P. Halmos, *Algebraic logic*, Chelsea Publishing Company, New York 1962.

[2] Z. Luo, *Clones and Genoids in Lambda Calculus and First Order Logic*, preprint, arXiv:0712.3088v2.

[3] Z. Luo, *Clone Theory: Its Syntax and Semantics, Applications to Universal Algebra, Lambda Calculus and Algebraic Logic*, preprint, arXiv:0810.3162.