Clone Theory
Its Syntax and Semantics
Applications to Universal Algebra, Lambda Calculus and Algebraic Logic

Zhaohua Luo

July 2007

(Revised October 2008)

Abstract

The primary goal of this paper is to present a unified way to transform the syntax of a logic system into certain initial algebraic structure so that it can be studied algebraically. The algebraic structures which one may choose for this purpose are various clones over a full subcategory of a category. We show that the syntax of equational logic, lambda calculus and first order logic can be represented as clones or right algebras of clones over the set of positive integers. The semantics is then represented by structures derived from left algebras of these clones.

Key words: Clone, Universal Algebra, First-Order Theory, Lambda Calculus, Polyadic Algebra

Introduction

Let $\mathbb{N} = \{1, 2, 3, ...\}$ be the set of positive integers. Denote by **Set** the category of sets.

We will introduce the following fundamental structures for universal algebra, lambda calculus and first order logic:
1. Clones over $\mathbb{N}$.
2. Left and right algebras of a clone over $\mathbb{N}$.
3. λ-clones, λ_β-clones (reflexive clones).

Email address: `zluo@azd.com` (Zhaohua Luo).

4. λ-algebras.
5. Predicate algebras with terms in a clone over $\mathbb{N}$.
6. Quantifier algebras with terms in a clone over $\mathbb{N}$.

Note that the class of objects in each category is a variety in the sense of universal algebra. Four basic observations on algebraization are:
(i) Finitary endofunctors of **Set** are represented by locally finitary right algebras of the initial clone over $\mathbb{N}$.
(ii) Finitary monads in **Set** (or finitary varieties) are represented by locally finitary clones over $\mathbb{N}$.
(iii) The set of λ-terms is represented by the initial λ-clone.
(iv) The set of formulas of a first order language is represented by a predicate algebra over the clone of terms.

The theory of clones considered in this paper originated from the theory of monads. Two equivalent definitions of monads, namely *monads in clone form* and *monads in extension form* given by E. Mane [11], can be interpreted as only defined over a given subcategory of the category. These are *clones* and *clones in extension form over a full subcategory*. It turns out that these two new concepts are no longer equivalent unless the subcategory is dense. But morphisms of clones, algebras of clones, and morphisms of algebras can all be defined for these two types of clones. Since many familiar algebraic structures, such as monoids, unitary Menger algebras, Lawvere theories, countable Lawvere theories, classical and abstract clones are all special cases of clones over various dense subcategories of **Set**, the syntax and semantics of these algebraic structures can be developed in a unified way, so that it is much easier to extend these results to many-sorted sets.

Just as monads arise from adjunctions of categories, clones arise from "adjunctions" of *species*. The notions of clones and species introduced in this paper are very easy to manipulate, yet they are more flexible than the traditional notions of monads and adjunctions.

Whenever it is convenient in this paper composition of morphisms in a category is written from left to right.

A *species* $\mathbf{N}/\mathbf{C}$ consists of a category $\mathbf{C}$ and a full subcategory $\mathbf{N}$ of $\mathbf{C}$; if $\mathbf{N} = \mathbf{C}$ then we say that $\mathbf{N}/\mathbf{N}$ (or simply $\mathbf{N}$) is a *singular species*. If $\mathbf{N}'/\mathbf{C}'$ is another species, a *function* (resp. *functor*) T from $\mathbf{N}'/\mathbf{C}'$ to $\mathbf{N}/\mathbf{C}$ is a function $T : Ob\mathbf{C}' \to Ob\mathbf{C}$ (resp. functor $T : \mathbf{C}' \to \mathbf{C}$) such that
(i). $Ob\mathbf{N}' = Ob\mathbf{N}$.
(ii). $\mathbf{C}'(A, B) = \mathbf{C}(A, TB)$ for $A \in \mathbf{N}$ and $B \in \mathbf{C}'$.
(iii). $r(fg) = (rf)g$ (resp. $f(Tg) = fg$) for $C, D \in \mathbf{N}$, $r \in \mathbf{N}(D, C)$, $f \in \mathbf{C}'(C, A)$ and $g \in \mathbf{C}'(A, B)$.
Note that the composite of two functions (or functors) of species is a function

(or functor) of species.

A *clone theory* (resp. *clone theory in extension form*) over a full subcategory $\mathbf{N}$ of a category $\mathbf{C}$ is a pair $(\mathbf{N}', T)$ where $\mathbf{N}'$ is a category and T is a function (resp. functor) of species from $\mathbf{N}'/\mathbf{N}'$ to $\mathbf{N}/\mathbf{C}$. We often simply say that $\mathbf{N}'$ or T is a clone theory over $\mathbf{N}$.

Suppose $\mathbf{N}/\mathbf{C}$ is a species. A *species over* $\mathbf{N}/\mathbf{C}$ is a pair $(\mathbf{N}'/\mathbf{C}', T)$ consists of a species $\mathbf{N}'/\mathbf{C}'$ and a functor $T : \mathbf{N}'/\mathbf{C}' \to \mathbf{N}/\mathbf{C}$.

If $(\mathbf{N}'/\mathbf{C}', T)$ is a species over $\mathbf{N}/\mathbf{C}$ then $(\mathbf{N}', T|_{\mathbf{N}'})$ is a clone theory over $\mathbf{N}$, called the *clone theory of* $\mathbf{N}'/\mathbf{C}'$, denoted by $Clone((\mathbf{N}'/\mathbf{C}'), T)$. Note that $\mathbf{N}'$ consists of free objects of $\mathbf{C}'$ over $\mathbf{N}$ with respect to $T : \mathbf{C}' \to \mathbf{C}$.

Let $(\mathbf{N}', T)$ be a clone theory in extension form over $\mathbf{N}$. By an $(\mathbf{N}', T)$*-species* we mean a species over $\mathbf{N}/\mathbf{C}$ with $(\mathbf{N}', T)$ as its clone theory.

The class $Sp(\mathbf{N}', T)$ of $(\mathbf{N}', T)$-species viewed as concrete categories over $\mathbf{C}$ forms a meta-category. Clearly $(\mathbf{N}', T)$ is the initial object of $Sp(\mathbf{N}', T)$. $Sp(\mathbf{N}', T)$ has a canonical terminal object $\mathbf{C}^T$, called the Eilenberg-Moore species of $(\mathbf{N}', T)$, consisting of T-algebras (see Definition 12).

Remark 1 *1. A functor from a singular species to a species is a clone theory in extension form.*

2. A functor from a species to a singular species is equivalent to an adjunction of categories.

3. A functor from a singular species to a singular species is equivalent to a monad.

4. Any functor from a species to another species determines a clones theory in extension form.

If $\mathbf{N}$ is dense in $\mathbf{C}$ then the notion of a clone theory over $\mathbf{N}$ is equivalent to that of a clone theory over $\mathbf{N}$ in extension form. In practice one is primarily interested in the clone theories over a dense subcategory. For this reason if $\mathbf{N}$ is not dense in $\mathbf{C}$ we always choose a proper subcategory $\mathbf{C}'$ of $\mathbf{C}$ containing $\mathbf{N}$ such that $\mathbf{N}$ is dense in $\mathbf{C}'$. Hence the distinction between the two types of clone theories is not essential.

Since a clone theory (or clone theory in extension form) over $\mathbf{N}$ gets objects and morphisms from $\mathbf{C}$, they can be intrinsically defined as algebraic systems in $\mathbf{C}$, called *clone systems*. The study of clone systems is presented in Section 1. Since clone theory and clone system are essentially the same, they are often referred just as a clone.

If $(\mathbf{N}', T)$ is a clone over $\mathbf{N}$, then there is a functor $F^T : \mathbf{N} \to \mathbf{N}'$ preserving objects and sending each $f : A \to B$ in $\mathbf{N}$ to $f\eta : A \to TB$ in $\mathbf{N}'$, where $\eta : B \to TB$ is the identity of $B \in \mathbf{N}'$. Linton in [7] defines a more general notion of a clone over a subcategory $\mathbf{N}$ of a category as an arbitrary functor $\mathbf{N} \to \mathbf{N}'$ which is bijective on objects. Our definition of algebras for a clone theory coincides with that of Linton's under functorial consideration. Thus all the results of [7] apply to clone theories. We mention that all the examples given in [7] p.22 are in fact clones over dense subcategories.

Example 0.1 *Examples of clones over dense subcategories are abundant. Here are some of the most elementary examples.*
1. A clone over a singleton in **Set** *is a monoid.*
2. A clone over a nonempty set N in **Set** *is a unitary Menger algebra T of rank $|N|$.*
3. A clone (resp. the dual of the clone theory) over the full subcategory $\{0, 1, 2, 3, ...\}$ *of* **Set** *is a clone in the classical sense (resp. Lawvere theory) (note that here each integer $n \geq 0$ is viewed as a finite set with n elements).*
4. The dual of a clone theory over the full subcategory $\{0, 1, 2, 3, ..., \mathbb{N}\}$ *(or* $(0, 1, \mathbb{N})$*) of* **Set** *is a countable Lawvere theory in the sense of [17].*
5. The dual of a clone theory over the full subcategory $\{1, 2, 3, ..., \mathbb{N}\}$ *(or* $(1, \mathbb{N})$*) is equivalent to an algebraic theory (see section 3).*
6. A clone over a one-object-category is a Kleisli algebra. Any clone over an infinite set N in **Set** *defines a Kleisli algebra (see section 5).*
7. Other important examples are clones over the subcategory of finitely presentable objects of a locally finitely presentable category.
8. A clone (or clone in extension form) over $\mathbf{N} = \mathbf{C}$ *is equivalent to a monad in extension form, or a Kleisli triple in* $\mathbf{C}$ *in the usual sense. Hence the notion of a clone generalizes that of a monad.*

The simplest type of clones are clones over an object N of a category $\mathbf{C}$. Recall that a *left act of a monoid M* (or a *left M-act*) is a set D together with a map $M \times D \to D$ such that $ed = d$ where e is the unit of M and $m(m'd) = (mm')d$ for any $m, m' \in M$ and $d \in D$. A *right act of M* is defined similarly.

Remark 2 *A clone over an object N of a category $\mathbf{C}$ is an object $\mathcal{A}$ of $\mathbf{C}$ such that $hom(N, \mathcal{A})$ is a monoid and $r(fg) = (rf)g$ for all $r : N \to N$ and $f, g : N \to \mathcal{A}$. Suppose $\mathcal{A}$ is a clone over N. A left $\mathcal{A}$-algebra is an object D such that $hom(N, D)$ is a left act of the monoid $hom(N, \mathcal{A})$ and $r(fm) = (rf)m$ for all $r : N \to N$, $f : N \to \mathcal{A}$ and $g : N \to D$. A right $\mathcal{A}$-algebra is a right act of $hom(N, \mathcal{A})$.*

Remark 3 *A clone in extension form over an object N of a category $\mathbf{C}$ is an object $\mathcal{A}$ such that $hom(N, \mathcal{A})$ is a monoid together with a homomorphism $T : hom(N, \mathcal{A}) \to hom(\mathcal{A}, \mathcal{A})$ of monoids such that $f(Tg) = fg$ for $f, g : N \to \mathcal{A}$. A left $\mathcal{A}$-algebra is then an object D such that $hom(N, D)$ is a left act of* *the monoid $hom(N,\mathcal{A})$ together with a homomorphism $H : hom(N,D) \to hom(\mathcal{A},D)$ of left acts of $hom(N,\mathcal{A})$ such that $f(Hg) = fg$ for $f : N \to \mathcal{A}$ and $g : N \to D$. A right $\mathcal{A}$-algebra is a right act of $hom(N,\mathcal{A})$.*

Assume $\mathbf{C} = \mathbf{Set}^S$ for a set of sorts S, and $N = \{N_s\}_{s\in S}$ is an S-sorted set. Then a clone in extension form over N is an S-sorted set $\mathcal{A} = \{\mathcal{A}_s\}_{s\in S}$ such that $hom(N,\mathcal{A})$ is a monoid and there is a homomorphism $T : hom(N,\mathcal{A}) \to hom(\mathcal{A},\mathcal{A})$ of monoids such that $f(Tg) = fg$. Since

$$hom(\mathcal{A},\mathcal{A}) = \prod_{s\in S} hom(\mathcal{A}_s,\mathcal{A}_s),$$

T is uniquely determined by a sequence of maps $\mathcal{A}_s \times hom(N,\mathcal{A}) \to \mathcal{A}_s$. Thus algebraically a clone in extension form over N can be defined as follows:

Definition 4 *A clone in extension form over an S-sorted set N is an S-sorted set $\mathcal{A}$ together with maps $\{\mu_s : \mathcal{A}_s \times hom(N,\mathcal{A}) \to \mathcal{A}_s\}_{s\in S}$ and a map $x : N \to \mathcal{A}$ such that for any $a \in \mathcal{A}_s$ and $f = \{f_s\}, g = \{g_s\} \in hom(N,\mathcal{A})$:*
(i) $(af)g = a(fg)$ where $(fg)_{si} = f_{si}g$ for any $s \in S$, $i \in N_s$ and $f_{si} = f_s(i)$.
(ii) $x_{si}f = f_{si}$.
(iii) $ax = a$.

If $\mathcal{A}$ is a clone in extension form over an S-sorted set N then each $\mathcal{A}_s$ is a right $hom(N,\mathcal{A})$-act and $hom(N,\mathcal{A})$ is the product $\prod_{s\in S} hom(N_s,\mathcal{A}_s)$ of these right $hom(N,\mathcal{A})$-acts. Categorically a clone in extension form over N is a category with a dense object $\mathcal{A}^*$ and an S-indexed set of objects $\{\mathcal{A}_s\}_{s\in S}$ such that $\mathcal{A}^*$ is the product $\prod_{s\in S} \mathcal{A}_s^{N_s}$.

Let $\mathbf{Set}_*^S$ be the set N of S-sorted sets such that N_s is not empty for any $s \in S$. An S-sorted set N is dense in $\mathbf{Set}_*^S$ if each N_s has at least two elements, thus the notions of a clone and a clone in extension form over such N are equivalent.

Let $\mathbb{N}$ be the set of positive integers. Let $\mathbb{N}_S = \{\mathbb{N}\}_{s\in S}$. Then the above analysis applies to clones over $\mathbb{N}_S$. Let $\mathcal{A}$ be a clone over $\mathbb{N}_S$. A left $\mathcal{A}$-algebra has the following form:

Definition 5 *A left $\mathcal{A}$-algebra is an S-sorted set D together with maps $\{\mu_s : \mathcal{A}_s \times D^{\mathbb{N}} \to D_s\}_{s\in S}$ such that for any $a \in \mathcal{A}_s$, $f \in \mathcal{A}^{\mathbb{N}}$ and $g \in D^{\mathbb{N}}$:*
(i) $(af)g = a(fg)$ where $(fg)_{si} = f_{si}g$ for any $s \in S$ and $i \in N_s$.
(ii) $x_{si}g = g_{si}$.

Denote by $Rg(\mathcal{A})$ and $Lg(\mathcal{A})$ the categories of right and left $\mathcal{A}$-algebras respectively. Then $Rg(\mathcal{A})$ is a topos and $Lg(\mathcal{A})$ is an algebraic category over $\mathbf{Set}^S$.

A clone $\mathcal{A}$ over $\mathbb{N}_S$ is *locally finitary* if for any $a \in \mathcal{A}_s$ we have $ae = a$ for a map $e : \mathbb{N}_S \to \mathcal{A}$ such that $e(\mathbb{N}_S) \subseteq x(\mathbb{N}_S)$, $e(\mathbb{N}_S)$ has finite components and $ee = e$; $e(\mathbb{N}_S)$ is then called a *support of* a. The assertions 2 and 3 of the following main theorem for many-sorted clones extend a similar theorem for (one-sorted) clones over $\mathbb{N}$ due to W. D. Neumann [14]:

Theorem 6 *1. The full subcategory of locally finitary clones over $\mathbb{N}_S$ is a coreflective subcategory of the category of clones over $\mathbb{N}_S$.*
2. The class of left $\mathcal{A}$-algebras of a locally finitary clone $\mathcal{A}$ over $\mathbb{N}_S$ is a finitary S-sorted variety.
3. Conversely, any finitary S-sorted variety V as a concrete category over $\mathbf{Set}^S$ is equivalent to the category of left algebras of the clone determined by the free algebra of V on $\mathbb{N}_S$.

Suppose B is any right $\mathcal{A}$-algebra. In preparation for the definitions of λ-clones and predicate algebras with terms in $\mathcal{A}$ we need an explicit form for the exponent $B^{\mathcal{A}_s}$ in the cartesian closed category $Rg(\mathcal{A})$ with respect to the right $\mathcal{A}$-algebra $\mathcal{A}_s$. It is a crucial fact for our purpose that the underlying set of $B^{\mathcal{A}_s}$ can be identified with the set B itself, so that a homomorphism $B^{\mathcal{A}_s} \to B$ reduces to a unary operation $B \to B$ (which is not an endomorphism of right $\mathcal{A}$-algebras in general). This can be seen as follows.

Since $\mathcal{A}_s$ and B are two right acts of the monoid $\mathcal{A}^{\mathbb{N}_S}$, the right act $B^{\mathcal{A}_s}$ can be defined as $\hom(\mathcal{A}^{\mathbb{N}_S} \times \mathcal{A}_s, B)$ with action $f \to fu$ of $u : \mathbb{N}_S \to \mathcal{A}$ on $f : \mathcal{A}^{\mathbb{N}_S} \times \mathcal{A}_s \to B$ defined by $(fu)(u', b) = f(uu', b)$ for any $u' : \mathbb{N}_S \to \mathcal{A}$ and $b \in B$ (cf. [10], p.62, ex.5). Since $\mathbb{N}$ is infinite, $\mathcal{A}_s^{\mathbb{N}} \times \mathcal{A}_s$ is isomorphic to $\mathcal{A}_s^{\mathbb{N}}$ as right $\mathcal{A}$-algebras. Since $\mathcal{A}_s^{\mathbb{N}}$ is generated by $x_s = [x_{s1}, x_{s2}, ...]$, $\mathcal{A}_s^{\mathbb{N}} \times \mathcal{A}_s$ is generated by $x_{s1}^* = ([x_{s2}, x_{s3}, ...], x_{s1})$. Fixed such an isomorphism via x_{s1}^*. We obtain an isomorphism $\mathcal{A}^{\mathbb{N}_S} \times \mathcal{A}_s = \prod_{s \in S} \mathcal{A}_s^{\mathbb{N}_S} \times \mathcal{A}_s$ to $\mathcal{A}^{\mathbb{N}_S} = \prod_{s \in S} \mathcal{A}_s^{\mathbb{N}_S}$. Since $\mathcal{A}^{\mathbb{N}_S}$ is the free right act generated by the unit x, there is a bijective map $\hom(\mathcal{A}^{\mathbb{N}_S}, B) \to B$ which maps f to $f(x)$ for $f : \mathcal{A}^{\mathbb{N}_S} \to B$. Thus the underlying sets of the following right $\mathcal{A}^{\mathbb{N}_S}$-acts are bijective

$$B^{\mathcal{A}_s} = \hom(\mathcal{A}^{\mathbb{N}_S} \times \mathcal{A}_s, B) \cong \hom(\mathcal{A}^{\mathbb{N}_S}, B) \cong B.$$

Note that as right $\mathcal{A}$-algebra $B^{\mathcal{A}_s}$ and B are not isomorphic in general. In section 5 we will give a direct definition of $B^{\mathcal{A}_s}$ using B (for the one-sorted case). The extension for many-sorted cases is straightforward.

Formally one may just specify a type of unary operations on B corresponding to homomorphisms from $B^{\mathcal{A}_s}$ to B (under the identification of $B^{\mathcal{A}_s}$ with B via x_{s1}^*), called *an abstract binding operation (on x_{s1})*. Unlike classical binding operation which binds a specific variable, an abstract binding operation only reduces the size of the support of a finitary element of B by 0 or 1. Elementary properties of abstract binding operations (for the one-sorted case) are given in section 4. The importance of an abstract binding operations lies in the fact that the classical unary binding operations such as

$$\boldsymbol{\lambda} x_1, \boldsymbol{\lambda} x_2, \boldsymbol{\lambda} x_3, \boldsymbol{\lambda} x_4, ...$$

$$\forall_s x_1, \forall_s x_2, \forall_s x_3, \forall_s x_4, ...(s \in S)$$
$$\exists_s x_1, \exists_s x_2, \exists_s x_3, \exists_s x_4, ...(s \in S)$$

can be defined as derived operations from abstract binding operations: λ, $\forall_s$ and $\exists_s$, respectively, provided the substitutions of variables have been defined. Thus to define λ-terms or formulas for a first order language, one could take abstract binding operations (e.g. λ or $\forall_s$) as the basic operation, and define the other classical binding operations via substitutions, as λ-terms form a clone, and formulas of a first order language form a right algebra of the clone of terms. This is the ideal approach to solve the problems of "variable capture" caused by substitutions in lambda and predicate calculus: at one hand the troublesome process of renaming bound variables is push into the proper background, on the other hand the classical binding operations are still available, which are more readable than an abstract binding operations. This is probably well known for lambda calculus (cf. P. Aczel [1] 2.3.1).

Definition 7 *A λ-clone* is a clone $\mathcal{A}$ over $\mathbb{N}$ together with two homomorphisms $\mathcal{A}^2 \to \mathcal{A}$ and $\mathcal{A}^{\mathcal{A}} \to \mathcal{A}$ of right $\mathcal{A}$-algebras.

The class of λ-clones forms a (non-finitary) variety. Therefore free λ-clones over any give set (called "holes" in literature) exist. The initial λ-clone is precisely the set of λ-terms in de Bruijn's notation (or the set of classical λ-terms modulo α-conversions). A clone $\mathcal{A}$ over $\mathbb{N}$ is called *reflexive* if $\mathcal{A}^{\mathcal{A}}$ is a retract of $\mathcal{A}$, *extensional* if $\mathcal{A}^{\mathcal{A}}$ is isomorphic to $\mathcal{A}$. One can show that a clone is reflexive (resp. extensional) if and only if it satisfies the axiom for β-conversion (resp. both β and η-conversions). The class of reflexive clones also forms a variety. The left algebras of the initial reflexive clone are *λ-algebras* considered in literature(cf. [3]).

Definition 8 *An S-sorted predicate algebra with terms in an S-sorted clone $\mathcal{A}$ over* $\mathbb{N}_S$ is a right $\mathcal{A}$-algebra P with the following homomorphisms:
1. $\Rightarrow: P^2 \to P$,
2. $\mathsf{F} : P^0 \to P$.
3. $\forall_s : P^{\mathcal{A}_s} \to P$ for each $s \in S$.
We also assume that there are identities $\approx_s \in P$ for each $s \in S$, where $\approx_s$ has support $\{x_{s1}, x_{x2}\}$.

Any left $\mathcal{A}$-algebra D determines a predicate algebra $P(D^{\mathbb{N}_S})$ where $P(D^{\mathbb{N}_S})$ is the power set of $D^{\mathbb{N}_S}$. A *model of a predicate algebra* P is then a pair $(D^{\mathbb{N}_S}, \mu)$ where D is a left $\mathcal{A}$-algebra and $\mu : P \to P(D^{\mathbb{N}_S})$ is a homomorphism of predicate algebras. An element $p \in P$ is *logic valid* if $\mu(p) = \mathsf{T}$ for any model of P, where $\mathsf{T} = (\mathsf{F} \Rightarrow \mathsf{F})$. For an S-sorted first-order language $\mathcal{L}$ the set $T(\mathcal{L})$

of terms of $\mathcal{L}$ is naturally a clone over $\mathbb{N}_S$, and the set $F(\mathcal{L})$ of formulas of $\mathcal{L}$ is a right $T(\mathcal{L})$-algebra. which is an S-sorted predicate algebra with terms in $T(\mathcal{L})$. The proof theory and model theory of $\mathcal{L}$ can then be carried out by algebraic considerations for the predicate algebra $F(\mathcal{L})$ and its models.

The paper is organized as follows. Section 1 consists of formal definitions of clones and left algebras of clones over a subcategory. In section 2 we study various properties of clones over $\mathbb{N}$. As applications of clone theory to universal algebra we discuss briefly how to define the notions of hyperidentities and hypervarieties in terms of C-clones. Also a purely algebraic approach to Morita theory for finitary varieties (i.e. locally finitary clones) is sketched at the end of the section. In section 3 we introduce a new type of algebraic theories consisting of only two objects, which are equivalent to clones over $\mathbb{N}$. In section 4 a general theory of binding unary operations on a right algebra of a clone is introduced. It is used to define λ-clones in section 5. Section 6 contains a simple modified classical approach to one-sorted first-order theory, using De Bruijin style formulas to provide a better substitution theory. The notion of one-sorted predicate algebra is given in section 7.

1 Clones

Let $\mathbf{C}$ be a category. Let $\mathbf{N}$ be a full subcategory of $\mathbf{C}$.

Definition 9 *A clone over* $\mathbf{N}$ *is a system* $T = (T, \eta, *)$ *consisting of functions*
(a) $T : Ob\mathbf{N} \to Ob\mathbf{C}$,
(b) η *assigns to each object* A *in* $\mathbf{N}$ *a morphism* $\eta_A : A \to TA$,
(c) $*$ *assigns to each ordered triple* (A, B, C) *of objects of* $\mathbf{N}$ *a function*

$$* : \mathbf{C}(A, TB) \times \mathbf{C}(B, TC) \to \mathbf{C}(A, TC)$$

such that for any $r : A \to B$, $h : A \to TB$, $f : B \to TC$ *and* $g : C \to TD$ *with* $D \in \mathbf{N}$ *we have*
(i) $(h * f) * g = h * (f * g)$.
(ii) $(r\eta_B) * f = rf$.
(iii) $f * \eta_C = f$.
Let $T = (T, \eta, *)$ *and* $T' = (T', \eta, \circ)$ *be two clones over* $\mathbf{N}$. *For each object* A *of* $\mathbf{N}$ *let* $\rho_A : TA \to T'A$ *be a morphism. Then* ρ *is a morphism of clones over* $\mathbf{N}$ *if it preserves* η *and* $(f * g)\rho_C = (f\rho_B) \circ (g\rho_C)$ *for any* $f : A \to TB$ *and* $g : B \to TC$.

Definition 10 *Suppose* T *is a clone over* $\mathbf{N}$. *A left* T*-algebra is a pair* $X = (X, \circ)$ *consisting of an object* X *of* $\mathbf{C}$ *and a function* $\circ$ *which assigns to each ordered pair* (A, B) *of objects of* $\mathbf{N}$ *a function* $\mathbf{C}(A, TB) \times \mathbf{C}(B, X) \to \mathbf{C}(A, X)$

such that for any $k : A \to C$, $f : A \to TB$, $g : B \to TC$ and $h : C \to X$ with $D \in \mathbf{N}$ we have
(i) $(f \circ g) \circ h = f \circ (g \circ h)$.
(ii) $(k\eta_C) \circ h = kh$.
Let X and Y be two T-algebras. A morphism of left T-algebras from X to Y is a morphism $\phi : X \to Y$ such that $(f \circ m)\phi = f \circ (m\phi)$ for any $f : A \to TB$ and $m : B \to X$.

Definition 11 *A clone over $\mathbf{N}$ in extension form is a system $T = (T, \eta, *-)$ consisting of functions*
(a) $T : Ob\mathbf{N} \to Ob\mathbf{C}$,
(b) η assigns to each object A in $\mathbf{N}$ a morphism $\eta_A : A \to TA$,
(c) $*-$ maps each morphism $f : B \to TC$ with B, C in $\mathbf{N}$ to a morphism $*f : TB \to TC$, such that for $g : C \to TD$ with $D \in \mathbf{N}$
(i) $*f * g = *(f * g)$.
(ii) $\eta_B * f = f$.
(iii) $*\eta_C = id_{TC}$.
Let T and T' be two clones in extension form over $\mathbf{N}$. For each object A of $\mathbf{N}$ let $\rho_A : TA \to T'A$ be a morphism. Then ρ is a morphism of clones if it preserves η and $*f\rho_B = \rho_A * (f\rho_B)$ for any $f : A \to TB$ with $B \in \mathbf{N}$.

Definition 12 *Suppose T is a clone in extension form over $\mathbf{N}$. A left T-algebra is a pair $X = (X, *-)$ consisting of an object X of $\mathbf{C}$ and a function $*-$ which maps each morphism $m : A \to X$ ($A \in \mathbf{N}$) to a morphism $*m : TA \to X$, such that*
(i) $(*g)(*m) = *(g * m)$ *for any $g : B \to TA$ with $B \in \mathbf{N}$.*
(ii) $\eta_X * m = m$.
Let X and Y be two left T-algebras. A morphism of left T-algebras from X to Y is a morphism $\phi : X \to Y$ such that $(*m)\phi = *(m\phi)$ for any $m : A \to X$ with $A \in \mathbf{N}$.

The class $\mathbf{C}^T$ of left T-algebras of a clone T in extension form over $\mathbf{N}$ is a concrete category over $\mathbf{C}$ with the forgetful functor $G^T : \mathbf{C}^T \to \mathbf{C}$. On the other direction we have a free functor $F^T : \mathbf{N} \to \mathbf{C}^T$ defined by $F^T(A) = (TA, *-)$ and $F^T(f) = f\eta_A$. If $\mathbf{N} = \mathbf{C}$ then (F^T, G^T) is an adjunction.

Proposition 13 *(cf. [7]) 1. $G^T : \mathbf{C}^T \to \mathbf{C}$ creates limits.*
2. $G^T : \mathbf{C}^T \to \mathbf{C}$ creates G^T-split coequalizers.
3. $F^T : \mathbf{N} \to \mathbf{C}_T$ preserves any colimit in $\mathbf{N}$ which is also a colimit in $\mathbf{C}$.
4. There is a bijection between morphisms $T \to T'$ of clones in extension form on $\mathbf{N}$ and functors $\mathbf{C}^{T'} \to \mathbf{C}^T$ of concrete categories over $\mathbf{C}$.

Definition 14 *Suppose T and T' are two clones in extension form over two full subcategories $\mathbf{N}$ and $\mathbf{N}'$ of $\mathbf{C}$ respectively.*
1. T and T' are rational equivalent if X^T and $X^{T'}$ are equivalent as concrete

category over $\mathbf{C}$.
2. T *and* T' *are Morita equivalent if* X^T *and* $X^{T'}$ *are equivalent as abstract category.*

Lemma 15 *Suppose* $\mathbf{N} \subset \mathbf{N}'$ *and any object of* $\mathbf{N}'$ *is a retract of an object of* $\mathbf{N}$*. Then any clone over* $\mathbf{N}'$ *in extension form is rationally equivalent to its restriction to* $\mathbf{N}$.

Lemma 16 *(i) Any clone* T *in extension form over* $\mathbf{N}$ *determines a clone over* $\mathbf{N}$*, called the clone of* T.
(ii) Any clone T *over a dense subcategory* $\mathbf{N}$ *defines a clone in extension form over* $\mathbf{N}$ *whose clone is* T.

Remark 17 *Suppose* $\mathbf{C}$ *is a complete category. Let* A *and* N *be two objects of* $\mathbf{C}$*. Let* A^{A^N} *be the* A^N*-th power of* A*. For any* $\nu : N \to A$ *let* $\diamond f : A^{A^N} \to A$ *be the projection determined by* ν*. Consider the map* $*- : (A^{A^N})^N \to (A^{A^N})^{(A^{A^N})}$ *defined by* $(*\alpha)(\diamond\nu) = \diamond(\alpha(\diamond\nu))$ *for any* $\alpha : N \to A^{A^N}$ *and* $\nu \in A^N$*. Let* $\eta : N \to A^{A^N}$ *be the map such that* $\eta(\diamond\nu) = \nu$ *for any* $\nu : N \to A$*. Then* $(A^{A^N}, \eta, *-)$ *is a clone over* N *in extension form, called a transformation monad.* Note that $(A, \diamond-)$ is an algebra of the clone $(A^{A^N}, *-)$. Let D be an object. Suppose $(A, \eta, *-)$ is a clone over N in extension form and $\theta : A \to D^{D^N}$ is a morphism. If $(D, (\theta\diamond)-)$ is an A-algebra then $\theta : (A, *-) \to (D^{D^N}, *-)$ is a morphism of clones over N. Conversely if $\theta : (A, *-) \to (D^{D^N}, *-)$ is a morphism of clones over N then $(D, (\theta\diamond)-)$ is an A-algebra. It follows that if $\mathbf{C}$ is complete then an A-algebra can also be defined as an object D together with a morphism of clones over N from A to D^{D^N}.

Theorem 18 *(Cayley's theorem for clones) Any clone* A *in extension form over an object* N *of a complete category is a subclone of the transformation clone* A^{A^N} *over* N.

2 Clones in Universal Algebra

In abstract universal algebra one studies clones over the set $\mathbb{N}$ of positive integers. Since $\mathbb{N}$ is dense in **Set**, the notion of clones and clones in extension form over $\mathbb{N}$ are equivalent. Such a clone can be defined in many different ways (see W. D. Neumann [14], B. M. Schein, V. S. Trohimenko [18], B. Pareigis and H. Rohrl [15], and Z. Luo [8] [9]). The following two equivalent definitions stand out as the most convenient definitions to use in practice:

If A is any set denote by $A^{\mathbb{N}}$ the set of infinite sequences $[a_1, a_2, ...]$ of elements of A. If A, B, C are three sets and $\alpha : A \times B^{\mathbb{N}} \to C$ is a function we often simply write $a[b_1, b_2, ...]$ for $\alpha(a, [b_1, b_2, ...]) \in C$ for any $a \in A$ and $b_1, b_2, ... \in B$. We shall use these notations in the definitions of clones over $\mathbb{N}$ and left or right algebras of a clone over $\mathbb{N}$.

Definition 19 *A clone over $\mathbb{N}$ is a triple $(\mathcal{A}, X, \sigma)$ where $\mathcal{A}$ is a nonempty set, $X = \{x_1, x_2, ...\}$ is a subset of $\mathcal{A}$, and $\sigma : \mathcal{A} \times \mathcal{A}^{\mathbb{N}} \to \mathcal{A}$ is a map such that for any $a, a_1, a_2, ..., b_1, b_2, ... \in \mathcal{A}$ we have*
1. $(a[a_1, a_2, ...])[b_1, b_2, ...] = a[a_1[b_1.b_2, ...], a_2[b_1, b_2, ...], ..]$.
2. $a[x_1, x_2, ...] = a$.
3. $x_i[a_1, a_2, ...] = a_i$ for any $i \in \mathbb{N}$.

Definition 20 *A clone over $\mathbb{N}$ is a nonempty set $\mathcal{A}$ such that*
1. $\mathcal{A}^{\mathbb{N}}$ is a monoid with a unit $\tilde{x} = [x_1, x_2, ...]$.
2. $\mathcal{A}$ is a right $\mathcal{A}^{\mathbb{N}}$-act.
3. $x_i[a_1, a_2, ...] = a_i$ for any $\tilde{a} = [a_1, a_2, ...] \in \mathcal{A}^{\mathbb{N}}$ and $i \in \mathbb{N}$.

Remark 21 *According to Definition 19 a clone over $\mathbb{N}$ is an algebra with an $\mathbb{N}$-ary operation $\alpha : \mathcal{A}^{\mathbb{N}} \times \mathcal{A} = \mathcal{A}^{\mathbb{N}} \to \mathcal{A}$ and a countably infinite sequence of constants $\{x_1, x_2, ...\}$ satisfying the three axioms. Thus the class of clones over $\mathbb{N}$ forms a (non-finitary) variety.*

From now on by a clone we always mean a *clone over* $\mathbb{N}$. We shall write $[[a_1, a_2, a_3, ..., a_n]]$ for $[a_1, a_2, a_3, ..., a_n, a_n, a_n, ...]$ for any $a_1, a_2, ..., a_n \in \mathcal{A}$. Let $X = \{x_1, x_2, ...\}$.

The notions of left and right algebras of clones on $\mathbb{N}$ defined below were first introduced in [15].

Definition 22 *Let $\mathcal{A}$ be a clone. A left algebra of $\mathcal{A}$ (or a left $\mathcal{A}$-algebra) is a set D together with a multiplication $\mathcal{A} \times D^{\mathbb{N}} \to D$ such that for any $a \in \mathcal{A}$, $[a_1, a_2, ...] \in \mathcal{A}^{\mathbb{N}}$ and $[d_1, d_2, ...] \in D^{\mathbb{N}}$*
1. $(a[a_1, a_2, ...])[d_1, d_2, ...] = a([a_1[d_1, d_2, ...], a_2[d_1, d_2, ...], ...]$.
2. $x_i[d_1, d_2, ...] = d_i$.

Example 2.1 *1. $\mathbb{N}$ is a clone with the monoid $\mathbb{N}^{\mathbb{N}}$. It is the initial clone.*
2. Similarly $X = \{x_1, x_2, ...\}$ is an initial clone which is isomorphic to $\mathbb{N}$.
3. Let $\tau = \{n_i\}_{i \in I}$ be a type of algebras. Let $X = \{x_1, x_2, ...\}$ be a set of variables. The term algebra $\mathbf{T}_\tau(X)$ defined in universal algebra is a clone. The category of $\tau(X)$-algebra is equivalent to the category of left $\mathbf{T}_\tau(X)$-algebras.

Definition 23 *Let $\mathcal{A}$ be a clone. A right algebra of $\mathcal{A}$ (or a right $\mathcal{A}$-algebra) is a right act B of monoid $\mathcal{A}^{\mathbb{N}}$. Suppose $\mathcal{A}$ and $\mathcal{B}$ are two clones. An $\mathcal{A}$-$\mathcal{B}$-algebra is a left $\mathcal{A}$-algebra and right $\mathcal{B}$-algebra B such that $(a\tilde{c})\tilde{b} = a(\tilde{c}\tilde{b})$ for any $a \in \mathcal{A}$, $\tilde{c} \in B^{\mathbb{N}}$ and $\tilde{b} \in \mathcal{B}^{\mathbb{N}}$.*

Example 2.2 *$\mathcal{A}$ is an $\mathcal{A} - \mathcal{A}$-algebra.*

We say an element of b of a right $\mathcal{A}$-algebra B has *finite rank* $n > 0$ if $b[[x_1, x_2, ..., x_n]] = b$. We say b has *rank* 0 (or b *is closed*, or b *is a sentence*) if $b[[x_1]] = b[[x_2]] = b$. An element $b \in B$ has rank $n \geq 0$ if and only if the left translation $l_a : \mathcal{A}^{\mathbb{N}} \rightarrow B$ with $\tilde{a} \rightarrow b\tilde{a}$ only depends on the first n components of $\tilde{a}$. Denote by $\mathcal{F}(B)$ the set of finitary elements of B. Denote by $\mathcal{F}_n(B)$ the set of elements of B with rank $n \geq 0$. B is *locally finitary* if any element of B has finite rank. We obtain a sequence of sets:

$$\mathcal{F}_0(B) \subseteq \mathcal{F}_1(B) \subseteq \mathcal{F}_2(B) \subseteq ...\mathcal{F}(B) \subseteq B.$$

Since $\mathcal{A}$ itself is also a right $\mathcal{A}$-algebra, these notions also apply to $\mathcal{A}$. Hence we have a sequence of subsets of $\mathcal{A}$:

$$\mathcal{F}_0(\mathcal{A}) \subseteq \mathcal{F}_1(\mathcal{A}) \subseteq \mathcal{F}_2(\mathcal{A}) \subseteq ...\mathcal{F}(\mathcal{A}) \subseteq \mathcal{A}.$$

Lemma 24 *1. $\mathcal{F}(\mathcal{A})$ is a locally finitary clone.*
2. The category of locally finitary clone is a full coreflectvie subcategory of the category of clones.
3. Each $\mathcal{F}_i(\mathcal{A})$ is a free left $\mathcal{A}$-algebra of rank $i \geq 0$.
4. $\mathcal{A}$ is a free left $\mathcal{A}$-algebra of countable rank.

Definition 25 *The dull $Law(\mathcal{A})$ of the full subcategory $\mathcal{F}_0(A) \subseteq \mathcal{F}_1(A) \subseteq \mathcal{F}_2(A) \subseteq ...$ of the category of left $\mathcal{A}$-algebras is called the Lawvere theory of $\mathcal{A}$. Note that $Law(\mathcal{A}) = Law(\mathcal{F}(\mathcal{A}))$.*

Lemma 26 *Suppose $\mathcal{A}$ is a locally finitary clone.*
1. If B is a right $\mathcal{A}$-algebra then $\mathcal{F}(\mathcal{B})$ is a locally finitary right $\mathcal{A}$-algebra.
2. The category of locally finitary right $\mathcal{A}$-algebras is a full coreflectvie subcategory of right $\mathcal{A}$-algebras.

Theorem 27 *1. If $\mathcal{A}$ is a locally finitary clone then the class of left $\mathcal{A}$-algebras is a finitary variety. Conversely any finitary variety arises in this way.*
2. The category of locally finitary clone is equivalent to the opposite of the category of finitary varieties (as concrete categories over **Set***).*
3. The category of locally finitary clone is equivalent to the category of Lawvere algebraic theories (without terminate object).
4. The category of locally finitary clone is equivalent to the category of finitary monads in **Set***.*

Definition 28 *Let V be a variety of type τ. Let $\mathcal{A}$ be a clone. An $(\mathcal{A}, V)$-algebra is a set B which is a right $\mathcal{A}$-algebra and an algebra in V such that for any n-ary operation symbol f in τ the operation $f^B : B^n \rightarrow B$ is a homomorphism of right $\mathcal{A}$-algebras, or equivalently, each action on B induces an endomorphism on B. If $B = \mathcal{A}$ then we say $\mathcal{A}$ is a V-clone. If V is the variety of all τ-algebras then an $(\mathcal{A}, V)$-algebra or V-clone is called an $(\mathcal{A}, \tau)$-algebra or τ-clone respectively. Denote by Clone the variety of clones. A Clone-clone* *is called a C-clone.*

Definition 29 *A V-clone (or τ-clone) $\mathcal{A}$ is called primary if the algebra $\mathcal{A}$ in V is generated by X.*

Lemma 30 *1. If $\mathcal{A}$ is a primary τ-clone then the class of left $\mathcal{A}$-algebras is a τ-variety with $\mathcal{A}$ as the free algebra over X.*

2. The category of primary τ-clones is equivalent to the opposite of the category of τ-varieties.

Example 2.3 *1. Suppose T and D are two left algebras of clones $\mathcal{A}$ and $\mathcal{B}$ respectively. Then the set of maps $T^{D^{\mathbb{N}}}$ from $D^{\mathbb{N}}$ to T is an $\mathcal{A}-\mathcal{B}$-algebra.*
2. Suppose D is a left $\mathcal{A}$-algebra and T is an algebra in a variety V, then $T^{D^{\mathbb{N}}}$ is an $(\mathcal{A}, V)$-algebra.

Example 2.4 *Suppose B is an algebra in a variety V. Then the basic operations on B extend to $B^{B^{\mathbb{N}}}$ point-wisely, so $B^{B^{\mathbb{N}}}$ is a V-clone. Denote by $Cl(B)$ the subalgebra of $B^{B^{\mathbb{N}}}$ generated by the projections $\pi_1, \pi_2, ...$ from $B^{\mathbb{N}}$ to B. One can show by induction that $Cl(B)$ is a subclone of $B^{B^{\mathbb{N}}}$, thus it is the smallest sub-V-clone of $B^{B^{\mathbb{N}}}$. If V is a finitary variety then $Cl(B)$ is a locally finitary clone, and the Lawvere theory $Law(Cl(B))$ of $Cl(B)$ determines the clone of B in the classical sense of P. Hall (cf. [4] p.126). If B is a clone then $Cl(B) \subseteq B^{B^{\mathbb{N}}}$ are C-clones.*

A C-clone can also be defined directly:

Definition 31 *A C-clone* is an algebra $\mathcal{H}$ with two $\mathbb{N}$-ary operations $\cdot, \; * : \mathcal{H} \times \mathcal{H}^{\mathbb{N}} = \mathcal{H}^{\mathbb{N}} \to \mathcal{H}$ and two sequences $t_1, t_2, ..., s_1, s_2, ... \in \mathcal{H}$ such that
1. $(\mathcal{H}, \; \cdot, \{t_1, t_2, ...\})$ and $(\mathcal{H}, *, \{s_1, s_2, ...\})$ are clones.
2. $(au) * v = (a * v)[u_1 * v, u_2 * v, ...]$ for any $a \in \mathcal{H}$ and $u, v \in \mathcal{H}^{\mathbb{N}}$.
3. $t_i * u = t_i$ for any $u \in \mathcal{H}^{\mathbb{N}}$.

The class of C-clones is a variety. Therefore the initial C-clone E exists. It is the free clone on $\{s_1, s_2, ...\}$. For any C-clone $\mathcal{H}$ the unique homomorphism $L_{\mathcal{H}} : E \to \mathcal{H}$ is not injective in general, which determines a congruence $Cng(\mathcal{H})$ of E. We say that *a C-clone $\mathcal{H}$ satisfies a hyperidentity $a \approx b$* if $< a, b > \in Cng(\mathcal{H})$. We say that *a clone $\mathcal{A}$ satisfies a hyperidentity $a \approx b$* if the C-clone $B^{B^{\mathbb{N}}}$ (or $Cl(\mathcal{A})$) satisfies the hyperidentity. We say that *a variety V satisfies a hyperidentity $a \approx b$* if the clone determined by the free algebra of V on $\{t_1, t_2, ...\}$ satisfies the hyperidentity. Thus we may speak of the hyperidentities satisfied by groups, rings, etc. The theory of hyperidentities are important for the study of classification of finitary varieties. A *hypervariety* is the totality of locally finitary clones satisfying a set of hyperidentities. A class of locally finitary clones is a hypervariety if and only if it is closed under subclones, homomorphic images and direct products of clones (cf. [5]).

Example 2.5 *(Post) Let n be any positive integer > 1 viewed as a finite set with n elements. Then $\mathcal{F}(n^{n^{\mathbb{N}}})$ is a locally finitary clone. The category of left $\mathcal{F}(n^{n^{\mathbb{N}}})$-algebras is equivalent to the category of boolean algebras. In particular, the category of left $\mathcal{F}(2^{2^{\mathbb{N}}})$-algebras is equivalent to the category of boolean algebras. Note that $\mathcal{F}(2^{2^{\mathbb{N}}})$ is naturally a free boolean algebra of countable rank.*

Example 2.6 *Let $\mathbb{F} = \{0, 1, 2, 3, ...\}$ and $\mathbb{F}' = \{1, 2, 3, ...\}$ be the full subcategories of* **Set**. *Then the category of locally finitary right X-algebras is equivalent to the category of presheaves in $\mathbf{Set}^{\mathbb{F}'}$. Since each presheaf in $\mathbf{Set}^{\mathbb{F}'}$ is the normalization of a presheaf in $\mathbf{Set}^{\mathbb{F}}$ at 0, one may replace presheaves in $\mathbf{Set}^{\mathbb{F}}$ by locally finitary right X-algebras (or more generally, by any right algebra of a clone) in the definition of binding algebras over variable sets given in [6].*

Definition 32 *A type is a right X-algebra.*

Denote by $Type$ (resp. $FType$) the category of types (resp. locally finitary types). Then $FType$ is a full coreflective subcategory of $Type$.

Suppose D and E are two locally finitary types. Then $E^{\mathbb{N}}$ is a bi-act of $\mathbb{N}^{\mathbb{N}}$. The tensor product $D \otimes E^{\mathbb{N}}$ is the quotient of the product type $D \times E^{\mathbb{N}}$ modulo the congruence generated by the relations $(d[y_1, y_2, ...], [e_1, e_2, ...]) \approx (d, [y_1, y_2, ...][e_1, e_2, ...])$ for all $d \in D$, $e_1, e_2, ... \in E$ and $y_1, y_2, ... \in X$. Then $D \otimes E^{\mathbb{N}}$ is a locally finitary type. One can show that $(FType, \otimes, X)$ is a strict monoidal category with the unit $\mathbb{N}$.

Any locally finitary type D determines a finitary endofunctor $\eta_D : \mathbf{Set} \to \mathbf{Set} : E \to D \otimes E$ (here each set E is viewed as a type with trivial actions). Conversely, if $F : \mathbf{Set} \to \mathbf{Set}$ is any finitary endofunctor then $F(\mathbb{N})$ is naturally a locally finitary type.

Theorem 33 *1. The category of strict monoidal category of locally finitary types is equivalent to the strict monoidal category of finitary endofunctors of* **Set** *with compositions as multiplications.*

2. The category of locally finitary clones is equivalent to the category of monoids in the monoidal category $(FType, \otimes, X)$.

Example 2.7 *If $\mathcal{A}$ is a clone and Y is any set then $\mathcal{A} \otimes Y$ is naturally a left $\mathcal{A}$-algebra which is the free left $\mathcal{A}$-algebra over Y.*

Theorem 34 *If $\mathcal{A}$ is any clone denote by $\mathcal{E}(\mathcal{A}^{\mathbb{N}})$ the set $\tilde{e}$ of idempotents of $\mathcal{A}^{\mathbb{N}}$ such that $\tilde{e} = [[x_1, x_2, ..., x_n]]\tilde{e}[[x_1, x_2, ..., x_n]]$ for some $n > 0$, which is viewed as a subcategory of Karoubi envelope of $\mathcal{A}^{\mathbb{N}}$ (cf. [3] p.114). Then two locally finitary clones $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent iff the categories $\mathcal{E}(\mathcal{A})$ and $\mathcal{E}(\mathcal{B})$ are equivalent (cf. [2]) (note that bi-algebras of clones can also be* *used to characterize Morita equivalent clones) .*

Suppose $\mathcal{A}$ is a clone. Let $n > 0$ be a positive integer. If $u = (a_1, ..., a_n), v = (b_1, ..., b_n) \in \mathcal{A}^n$ we write $u + v$ for the join $(a_1, ..., a_n, b_1, ..., b_n)$. The *n-th metrix power of* $\mathcal{A}$ is the clone $\mathcal{A}^{[n]}$ with $\mathcal{A}^n$ as the universe such that the new multiplication $\mathcal{A}^n \times (\mathcal{A}^n)^{\mathbb{N}} \to \mathcal{A}^n$ is defined by $u[v_1, v_2, ...](i) = u(i)[v_1 + v_2 + ...]$ for all $u, v_1, v_2, ... \in \mathcal{A}^n$ and $i \leq n$. The unit of $\mathcal{A}^{[n]}$ is

$$[(x_1, ..., x_n), (x_{n+1}, ..., x_{2n}), (x_{2n+1}, ..., x_{3n}), ...)].$$

Using the above theorem one can show that $\mathcal{A}$ and $\mathcal{A}^n$ are Morita equivalent for locally finitary $\mathcal{A}$.

There is yet another standard method to obtain Morita equivalent clones, called *modification*. Call an element $a \in \mathcal{A}$ a *varietal generator (of rank 1)* if $a = a[[x_1]] = a[[a]]$ and $[[x_1]] = \tilde{c}([[a]][[x_n]])\tilde{d}$ for some $\tilde{a}, \tilde{d} \in \mathcal{A}^{\mathbb{N}}$ and $n > 0$. If a is a varietal generator then $a\mathcal{A}^{\mathbb{N}}$ is a clone with $[a[[x_1]], a[[x_2]], ...]$ as the unit. Using the above theorem again one can show that $\mathcal{A}$ and $a\mathcal{A}^{\mathbb{N}}$ are Morita equivalent.

Theorem 35 *A locally finitary clone $\mathcal{B}$ is Morita equivalent to a locally finitary clone $\mathcal{A}$ iff $\mathcal{B} \cong a(\mathcal{A}^{[n]})^{\mathbb{N}}$ for a varietal generator a of $\mathcal{A}^{[n]}$ for some $n > 0$ (cf. [12]).*

There are many other interesting aspects of clone theory. Since Lawvere theories, finitary varieties and finitary monads over **Set** are essentially locally finitary clones over $\mathbb{N}$, any notion applies to Lawvere theories, finitary varieties and finitary monads over **Set** can be transformed into a purely algebraic notion for locally finitary clones (or their left or right algebras). Furthermore, if a notion does not specifically refer to the finiteness condition then it can also be extended to arbitrary clones. Here are some examples: tensor product of clones, commutative clone, Morita theory for clones, Malcev clone, arithmetical clone, minimal clone, discriminator clone, spectrum of a clone, Post algebra, cylindric algebra or polyadic algebra with terms in a clone, Fiore-Plotkin-Turi substitution algebra, and operads, etc.

3 Algebraic Theories

Definition 36 *1. An algebraic theory is a category $(T, T^{\mathbb{N}})$ of two objects such that $T^{\mathbb{N}}$ together with a map $x : \mathbb{N} \to \hom(T^{\mathbb{N}}, T)$ is the $\mathbb{N}$-th power of T.*
2. Suppose $(T, T^{\mathbb{N}})$ and $(H, H^{\mathbb{N}})$ are two algebraic theories. A functor $F : (T, T^{\mathbb{N}}) \to (H, H^{\mathbb{N}})$ is called a morphism of algebraic theories if F sends T to H, $T^{\mathbb{N}}$ to $H^{\mathbb{N}}$ and $F(x(i)) = x(i)$ for any $i \in \mathbb{N}$.
3. A left model of an algebraic theory $(T, T^{\mathbb{N}})$ is a functor $(T, T^{\mathbb{N}}) \to$ **Set**

preserving the $\mathbb{N}$*-th power of* T*. A homomorphism of left models of* $(T, T^{\mathbb{N}})$ *is a natural transformation.*
4. A right model of $(T, T^{\mathbb{N}})$ *is a functor* $(T, T^{\mathbb{N}})^{op} \to \mathbf{Set}$*. A homomorphism of right models of* $(T, T^{\mathbb{N}})$ *is a natural transformation.*

Remark 37 *1. Any clone* $\mathcal{A}$ *determines an algebraic theory* $(\mathcal{A}, \mathcal{A}^{\mathbb{N}})$*, which is the full subcategory of right acts of* $\mathcal{A}^{\mathbb{N}}$ *generated by the two right acts* $\mathcal{A}$ *and* $\mathcal{A}^{\mathbb{N}}$*. The algebraic theory* $(\mathcal{A}, \mathcal{A}^{\mathbb{N}})$ *is called a matrix algebraic theory.*
2. Any algebraic theory $(T, T^{\mathbb{N}})$ determines a clone $\hom(T^{\mathbb{N}}, T)$ with the monoid $\hom(T^{\mathbb{N}}, T)^{\mathbb{N}} = \hom(T^{\mathbb{N}}, T^{\mathbb{N}})$ and the unit $x : \mathbb{N} \to \hom(T^{\mathbb{N}}, T)$.
3. These processes are inverse to each other. Thus the notion of clones (over $\mathbb{N}$) is equivalent to the notion of algebraic theories. More precisely, we have the following

Lemma 38 *1. Any algebraic theory* $(T, T^{\mathbb{N}})$ *is isomorphic to the matrix algebraic theory* $(\hom(T^{\mathbb{N}}, T), \hom(T^{\mathbb{N}}, T)^{\mathbb{N}})$*.*
2. The category of clones (over $\mathbb{N}$*) is equivalent to the category of algebraic theories.*

Example 3.1 *Let* V *be a variety. Let* $F(1)$ *and* $F(\mathbb{N})$ *be the free algebras of rank* 1 *and* $\mathbb{N}$ *respectively. Then* $F(\mathbb{N})$ *is the* $\mathbb{N}$*-th sum of* $F(1)$*. Thus the dual of the subcategory* $(F(1), F(\mathbb{N}))$ *of* V *is an algebraic theory, and* $F(\mathbb{N}) = hom(F(1), F(\mathbb{N}))$ *is a clone, called the clone of* V. For instance, if S is any set then $(S, \mathbb{N} \times S)^{op}$ is an algebraic theory. Thus $(\mathbb{N} \times S)^S$ is a clone.

4 Binding Operations

Suppose $\mathcal{A}$ is a clone and B is a right $\mathcal{A}$-algebra. Denote by $\tilde{x} = [x_1, x_2, ...]$ the unit of the monoid $\mathcal{A}^{\mathbb{N}}$.

If $a \in \mathcal{A}$ and $\tilde{b} \in B^{\mathbb{N}}$ we let
$[+] = [x_2, x_3, x_4, x_5, ...] \in \mathcal{A}^{\mathbb{N}}$.
$[-] = [x_1, x_1, x_2, x_3, x_4, x_5, ...] \in \mathcal{A}^{\mathbb{N}}$.
$[+i] = [x_1, x_2, ..., x_{i-1}, x_{i+1}, ...] \in \mathcal{A}^{\mathbb{N}}$.
$[-i] = [x_1, x_2, ..., x_{i-1}, x_i, x_i, x_{i+1}, ...] \in \mathcal{A}^{\mathbb{N}}$.
$a^+ = a[+]$, $a^- = a[-]$, $a^{+i} = a[+i]$, $a^{-i} = a[-i]$, $\tilde{b}^{+i} = \tilde{b}[+i]$, $\tilde{b}^{-i} = \tilde{b}[-i]$.

In the following we assume $y, z, w, ... \in \{x_1, x_2, ...\}$, which are called *syntactical variables.* If $a, b \in B$ let $a[b/x_i] = a[x_1, x_2, ..., x_{i-1}, b, x_{i+1}, ...]$. We say $a \in B$ is *independent of a syntactical variable* y if $a = a[y^+/y]$; otherwise we say that a *depends on* y. Denote by $FV(a)$ the set of variables on which a depends. If a has rank $n \geq 0$ then $FV(a) \subseteq \{x_1, x_2, ..., x_n\}$. If a is closed then $FV(a) = \emptyset$. If a is finitary and $FV(a) = \emptyset$ then a is closed.

Definition 39 *A map $\lambda : B \to B$ is called an abstract binding operation on x_i* if
$$(\lambda b)\tilde{a} = \lambda(b[a_1^{+i}, a_2^{+i}, ..., a_{i-1}^{+i}, x_i, a_i^{+i}, ...])$$
for any $b \in B$ and $\tilde{a} \in A^{\mathbb{N}}$.

Definition 40 *An operation $\lambda : B \to B$ is called binding on x_i* if

$$(\lambda b)\tilde{a} = (\lambda(b[a_1^{+i}, a_2^{+i}, ..., a_{i-1}^{+i}, x_i, a_{i+1}^{+i}, ...]))^{[-i]}$$

for any $b \in B$ and $\tilde{a} \in A^{\mathbb{N}}$.

Lemma 41 *1. If $\lambda : B \to B$ is an abstract binding operation on x_i then the operation $\lambda^{+i} : B \to B$ sending b to $(\lambda b)^{+i}$ is binding on x_i.*
2. If $\lambda : B \to B$ is binding on x_i then the map $\lambda^{-i} : B \to B$ sending b to $(\lambda b)^{-i}$ is an abstract binding operation on x_i.
3. The set of abstract binding operations on x_i and the set of binding operations on x_i for B are bijective.

Remark 42 *1. A map $\lambda : B \to B$ is an abstract binding operation on x_1 if and only if $(\lambda b)\tilde{a} = (\lambda(b[x_1, a_1^+, a_2^+, ...]))$.*
2. A map $\lambda : B \to B$ is binding on x_1 if and only if $(\lambda b)\tilde{a} = (\lambda(b[x_1, a_2^+, a_3^+, ...]))^-$.

Definition 43 *Let $\lambda : B \to B$ be an abstract binding operation on x_1. For any variable $y = x_i$ we introduce a new map $\boldsymbol{\lambda} y : B \to B$ by*

$$\boldsymbol{\lambda} y.b = \lambda(b^+[x_1/y^+]) = \lambda(b[x_2, x_3, ..., y^-, y, x_1, y^{++}, ...])$$
$$= \boldsymbol{\lambda} x_i.b = \lambda(b^+[x_1/x_i^+]) = \lambda(b[x_2, x_3, ..., x_i, x_1, x_{i+2}, ...]).$$

Lemma 44 *If $j \le i$ then $\boldsymbol{\lambda} x_i.b = (\boldsymbol{\lambda} x_j.(b^{+j}[x_j/x_i^+]))^{-j}$.*
If $i < j$ then $\boldsymbol{\lambda} x_i.b = (\boldsymbol{\lambda} x_j.(b^{+j}[x_j/x_i]))^{-j}$.

Suppose $\lambda : B \to B$ is an abstract binding operation on x_1 and $b \in B$.

Lemma 45 *1. $\boldsymbol{\lambda} x_1.b = \lambda(b[x_1, x_3, x_4, ...]) = (\lambda b)^+$.*
2. $\lambda b = (\boldsymbol{\lambda} x_1.b)^-$.
3. $(\boldsymbol{\lambda} x_1.b)\tilde{a} = (\boldsymbol{\lambda} x_1.(b[x_1, a_2^+, a_3^+, ...]))^-$
4. $\boldsymbol{\lambda} x_1.b = (\boldsymbol{\lambda} x_1.(b[x_1, x_3, x_4, ...]))^-$.
5. $\boldsymbol{\lambda} x_i.b = (\boldsymbol{\lambda} x_1.(b[x_2, x_3, ..., x_{i-1}, x_i, x_1, x_{i+2}, ..]))^-$.
6. $\boldsymbol{\lambda} y.b = (\boldsymbol{\lambda} x_1.(b[x_2, x_3, ..., y^-, y, x_1, y^{++}, ..]))^-$.
7. $(\boldsymbol{\lambda} x_i.b)[u_1, u_2, ...] = \lambda(b[u_1^+, u_2^+, ..., u_{i-1}^+, x_1, u_{i+1}^+, u_{i+2}^+, ...])$.
8. $\boldsymbol{\lambda} y$ is binding on y.
9. $\lambda(\mathcal{F}_n(B)) \subset \mathcal{F}_{n-1}(B)$ for any $n > 0$.
10. $\lambda(\mathcal{F}_0(B)) \subset \mathcal{F}_0(B)$.
11. If $a \in B$ has finite rank $i > 0$ then $\lambda^i a$ is closed.

Lemma 46 *1. If b is independent of y then $\boldsymbol{\lambda} z.b = \boldsymbol{\lambda} y.(b[y/z])$.*
2. If b is independent of y then $\boldsymbol{\lambda} y.b = \lambda(b^+)$.

3. If $y \neq z$ and $a \in \mathcal{A}$ is independent of z then $(\boldsymbol{\lambda} z.b)[a/y] = \boldsymbol{\lambda} z.(b[a/y])$.
4. If b is independent of x_2 then $(\boldsymbol{\lambda} x_1.b)^- = \boldsymbol{\lambda} x_1.(b^-)$.
5. If $a_2, a_3, ...$ are independent of x_1 then $(\boldsymbol{\lambda} x_1.b)\tilde{a} = (\boldsymbol{\lambda} x_1.(b[x_1, a_2^+, a_3^+, ...]))^- = \boldsymbol{\lambda} x_1(b[x_1, a_2, a_3, ...])$.

Remark 47 *We have*
$(\boldsymbol{\lambda} x_1.x_1)^- = \boldsymbol{\lambda} x_1.x_1 = \lambda x_1$ *(closed),*
$(\boldsymbol{\lambda} x_1.x_2)^- = \boldsymbol{\lambda} x_i.x_1 = \lambda x_2$ *(rank 1) for $i > 2$.*
$(\boldsymbol{\lambda} x_1.x_3)^- = \boldsymbol{\lambda} x_1.x_2 = \lambda x_3$ *(rank 2),*
$(\boldsymbol{\lambda} x_1.x_4)^- = \boldsymbol{\lambda} x_1.x_3 = \lambda x_4$ *(rank 3),*
...
The irregularity for $(\boldsymbol{\lambda} x_1.x_2)^-$ is due to the fact that the substitution $[x_1, x_1, x_3, ...]$ replaces x_2 by x_1, while in $\boldsymbol{\lambda} x_1.x_2$ the variable x_2 is bound by x_1.

5 Clones in Lambda Calculus.

Suppose $\mathcal{A}$ is a clone. Let B be a right $\mathcal{A}$-algebra, i.e. B is a right act of the monoid $\mathcal{A}^{\mathbb{N}}$. We define a new right $\mathcal{A}$-algebra $B^{\mathcal{A}} = (B, *)$ with the new action $* : B \times \mathcal{A}^{\mathbb{N}} \to B$: $b * [a_1, a_2, ...] = b[x_1, a_1^+, a_2^+, ...]$. A map $\lambda : B^{\mathcal{A}} \to B$ is a homomorphism of right $\mathcal{A}$-algebras if and only if $(\lambda b)\tilde{a} = \lambda(b * \tilde{a}) = \lambda(b[x_1, a_1^+, a_2^+, ...])$ for any $\tilde{a} \in \mathcal{A}^{\mathbb{N}}$. Thus $\lambda : B \to B$ is a homomorphism $B^{\mathcal{A}} \to B$ if and only if it is an abstract binding operation on x_1. Let $ev_{\mathcal{A},B} : B^{\mathcal{A}} \times \mathcal{A} \to B$ be the homomorphism of right $\mathcal{A}$-algebras defined by $ev(b, a) = b[a, x_1, x_2, ...]$ for any $b \in B$ and $a \in \mathcal{A}$. The right $\mathcal{A}$-algebra $B^{\mathcal{A}}$ together with the homomorphism $ev_{\mathcal{A},B} : B^{\mathcal{A}} \times \mathcal{A} \to B$ is the exponent in the category of right $\mathcal{A}$-algebras. Specifically, this means that, for any $f : T \times \mathcal{A} \to B$ of homomorphism of right $\mathcal{A}$-algebras, there is a unique $\Lambda f : T \to B^{\mathcal{A}}$ (called the *curred version of* f) given by $(\Lambda f)t = f(t^+, x_1)$ such that $f = ev \circ (\Lambda f \times id_{\mathcal{A}})$, as we have $ev \circ (\Lambda f \times id_{\mathcal{A}})(t, a) = ev_{\mathcal{A},B}(f(t^+, x_1), a) = f(t^+, x_1)[a, x_1, x_2, ...] = f(t^+[a, x_1, x_2, ..], x_1[a, x_1, x_2, ...]) = f(t, a)$.

The most important property about a clone is that it is a *Kleisli algebra*, i.e. a monad over a category with only one object. Algebraically a Kleisli algebra is a set S together with two monoid structures $(S, \cdot)$ and $(S, \circ)$ such that $a(b \circ c) = (ab) \circ c$ for any $a, b, c \in S$ (cf [11] p. 110. ex.18 and p.136. ex 5). Suppose $\mathcal{A}$ is a clone. $\mathcal{A}^{\mathbb{N}}$ carries another monoid structure with the binary operation $\circ$ in $\mathcal{A}^{\mathbb{N}}$ defined by

$$\tilde{a} \circ \tilde{b} = \tilde{a}[x_1, b_1, b_2, ...] = [a_1[x_1, b_1, b_2, ...], a_2[x_1, b_1, b_2, ...], ...],$$

whose unit is $[+] = [x_2, x_3, ...]$. Denote by $(\mathcal{A}^{\mathbb{N}}, \circ)$ this new monoid. There are three basic homomorphisms of monoids:

$$\Delta_1 : \mathcal{A}^{\mathbb{N}} \to (\mathcal{A}^{\mathbb{N}}, \circ) \quad [a_1, a_2, ...] \to [a_1^+, a_2^+, ...],$$

$$\Delta_2 : (\mathcal{A}^{\mathbb{N}}, \circ) \to \mathcal{A}^{\mathbb{N}} \ \ [a_1, a_2, ...] \to [x_1, a_1, a_2, ...].$$
$$\Delta = \Delta_2\Delta_1 : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}} \ \ [a_1, a_2, ...] \to [x_1, a_1^+, a_2^+, ...].$$

Then Δ_1 is a left adjoint of Δ_2, which induces a monad $(\Delta, [+], [x_1, x_1, x_2, ...])$ on the one object category $\mathcal{A}^{\mathbb{N}}$. We have $\tilde{a}(\tilde{b}\circ\tilde{c}) = (\tilde{a}\tilde{b})\circ\tilde{c}$ for $\tilde{a}, \tilde{b}, \tilde{c} \in \mathcal{A}^{\mathbb{N}}$. Thus $(\mathcal{A}, \cdot, \circ)$ is a Kleisli algebra. Denote by $Rg(\mathcal{A})$ the category of right $\mathcal{A}$-algebras. Let $\delta : Rg(\mathcal{A}) \to Rg(\mathcal{A})$ be the functor induced by the homomorphism $\Delta : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$. Then for any right $\mathcal{A}$-algebra B we have $\delta(B) = B^{\mathcal{A}}$.

Example 5.1 *The submonoid of $\mathbb{N}^{\mathbb{N}}$ generated by $[+]$ and $[-]$ is the initial Kleisli Algebra.*

Let $\mathbf{C}$ be a cartesian closed category. An object $U \in \mathbf{C}$ is *reflexive* if the exponent U^U is a retract of U, i.e. there are maps $F : U \to U^U$ and $G : U^U \to U$ such that $FG = id_{U^U}$.

Definition 48 *1. A reflexive clone* is a clone $\mathcal{A}$ together with two homomorphisms $\lambda : \mathcal{A}^{\mathcal{A}} \to \mathcal{A}$ and $\lambda^* : \mathcal{A} \to \mathcal{A}^{\mathcal{A}}$ such that $\lambda^*\lambda = id_{\mathcal{A}^{\mathcal{A}}}$.
2. An *extensional clone* is a clone $\mathcal{A}$ together with a bijective homomorphism (i.e. an isomorphism) from $\mathcal{A}$ to $\mathcal{A}^{\mathcal{A}}$ (cf [3]).

Definition 49 *(Elementary Definition) A reflexive clone* (resp. *extensional clone*) is a clone $\mathcal{A}$ together with two maps $\lambda, \lambda^* : \mathcal{A} \to \mathcal{A}$ such that
1. $\lambda^*(a\tilde{a}) = (\lambda^* a)[x_1, a_1^+, a_2^+, ...]$.
2. $(\lambda a)\tilde{a} = \lambda(a[x_1, a_1^+, a_2^+, ...])$ (resp. $\lambda\lambda^* = id_{\mathcal{A}}$).
3. $\lambda^*\lambda = id_{\mathcal{A}}$.

Lemma 50 *1. If $\mathcal{A}$ is a reflexive clone then $\mathcal{F}(\mathcal{A})$ is a locally finitary reflexive clone.*
2. The category of locally finitary reflexive clones is a coreflective subcategory of the category of reflexive clones.
3. The initial reflexive clone is locally finitary. The same is true for extensional clones.

Theorem 51 *Any reflexive object U in a cartesian closed category determines a reflexive clone.*

PROOF. 1. First assume the $\mathbb{N}$-th power $U^{\mathbb{N}}$ of U exists. Let $\mathcal{A} = \hom(U^{\mathbb{N}}, U)$. Since U^U is a retract of U and U is a retract of $U^{\mathbb{N}}$, $U^{\mathbb{N}}$ is dense in the subcategory $(U^U, U, U^{\mathbb{N}})$ of $\mathbf{C}$. Thus $(U^U, U, U^{\mathbb{N}})$ is equivalent to the category $(\mathcal{A}^{\mathcal{A}}, \mathcal{A}, \mathcal{A}^{\mathbb{N}})$ of right acts of $\mathcal{A}^{\mathbb{N}}$. Thus $\mathcal{A}$ is a reflexive right act.
2. If $U^{\mathbb{N}}$ dose not exist then one can embed the Lawvere theory $(U^0, U, U^2, U^3, ...)$ in the opposite of the category of its models. Then U is reflexive and the $\mathbb{N}$-th power $U^{\mathbb{N}}$ of U exists. Applying step 1 we obtain a reflexive clone.

Suppose $\lambda^* : \mathcal{A} \to \mathcal{A}^{\mathcal{A}}$ is a homomorphism. We have $\lambda^* a_1 = \lambda^*(x_1 \tilde{a}) = (\lambda^* x_1)[x_1, a_1^+, a_2^+, ...] = (\lambda^* x_1)[[x_1, a_1^+]]$. Thus $\lambda^* a = (\lambda^* x_1)[[x_1, a^+]]$ and $\lambda^*(x_1) \in \mathcal{F}_2(\mathcal{A})$. Define a homomorphism $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ of right $\mathcal{A}$-algebras by $ab = (\lambda^* x_1)[[b, a]]$. Then $\lambda^* a = a^+ x_1$. Conversely if $f : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is a homomorphism of right $\mathcal{A}$-modules then the map $\lambda^* : \mathcal{A} \to \mathcal{A}$ defined by $a \to a^+ x_1$ is a homomorphism from $\mathcal{A}$ to $\mathcal{A}^{\mathcal{A}}$.

Lemma 52 *The following three sets are bijective:*
1. $\hom(\mathcal{A}, \mathcal{A}^{\mathcal{A}})$.
2. The set of homomorphisms $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ of right $\mathcal{A}$-algebras.
3. $\mathcal{F}_2(\mathcal{A})$.

Definition 53 *A λ-clone is a clone with the following homomorphism of right $\mathcal{A}$-algebras:*
(i) $\lambda : \mathcal{A}^{\mathcal{A}} \to \mathcal{A}$.
(ii) An application $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ (or equivalently, a homomorphism $\lambda^* : \mathcal{A} \to \mathcal{A}^{\mathcal{A}}$).
A λ_β-clone is a λ-clone satisfying the following axiom for any $a \in \mathcal{A}$
(a) $(\lambda a)^+ x_1 = a$ (or equivalently, $\lambda^* \lambda = id_{\mathcal{A}}$) ($\beta$-conversion).
A $\lambda_{\beta\eta}$-clone is a λ_β-clone satisfying the following axiom for any $a \in \mathcal{A}$
(b) $\lambda(a^+ x_1) = a$ (or equivalently, $\lambda\lambda^* = id_{\mathcal{A}}$) ($\eta$-conversion).

The category of reflexive clones (resp. extensional clones) is equivalent to the category of λ_β-clones (resp. $\lambda_{\beta\eta}$-clones). The classes of λ-clones, λ_β-clones, and $\lambda_{\beta\eta}$-clones are varieties in the sense of universal algebra. Traditionally the initial λ-clone Λ is defined by λ-terms:

Definition 54 *(λ-terms) The class Λ of λ-terms is the least class satisfying the following*
1. *$x_i \in \Lambda$ for any $i \in \mathbb{N}$.*
2. *if $a \in \Lambda$ then $(\lambda a) \in \Lambda$*
3. *if $a, b \in \Lambda$ then $(ab) \in \Lambda$*

We define a multiplication $\Lambda \times \Lambda^{\mathbb{N}} \to \Lambda$ inductively:
1. $x_i[a_1, a_2, ...] = a_i$.
2. $(ab)[a_1, a_2, ...] = (a[a_1, a_2, ...])(b[a_1, a_2, ...])$.
3. $(\lambda a)[a_1, a_2, ...] = \lambda(a[x_1, a_1^+, a_2^+, ...])$.

One can prove by induction that $a[x_1, x_2, ...] = a$, and the following lemma

Lemma 55 Substitution Lemma. $(a\tilde{a})\tilde{b} = a[a_1\tilde{b}, a_2\tilde{b}, ...]$.

It is easy to see that Λ is a λ-clone. Clearly it is the initial λ-clone.

Definition 56 *1. Let Λ_β be the quotient of the λ-clone Λ by the congruence generated by all the pairs $\{< a, (\lambda a)^+ x_1 >\}_{a \in \Lambda}$. Λ_β is the initial λ_β-clone.*
2. Let $\Lambda_{\beta\eta}$ be the quotient of the λ-clone Λ by the congruence generated by

all the pairs $\{< a, (\lambda a)^+ x_1 >\}_{a\in\Lambda}$ *and* $\{< a, \lambda(a^+ x_1) >\}_{a\in\Lambda}$. *Then* $\Lambda_{\beta\eta}$ *is the initial* $\lambda_{\beta,\eta}$*-clone.*
(Note that it follows from Church-Rosser theorem that Λ_β *and* $\Lambda_{\beta\eta}$ *are not trivial (i.e. they contains more than one elements) [3]).*

Definition 57 *1. A left* Λ*-algebra is simply called a* Λ*-algebra.*
2. A left Λ_β*-algebra is called a* λ*-algebra.*
3. A left $\Lambda_{\beta\eta}$*-algebra is called an extensional* λ*-algebra.*
4. A left Λ_β*-algebra* D *is called a* λ*-model if the following axiom is satisfied: if* $a, b \in \mathcal{A}$ *and* $a\tilde{m} = b\tilde{m}$ *for every* $\tilde{m} \in D^{\mathbb{N}}$ *then* $\lambda a = \lambda b$.

Since Λ is finitary with Λ_β and $\Lambda_{\beta\eta}$ as quotients, these two classes are also finitary. Hence the classes of Λ-algebras (resp. λ-algebras, resp. extensional λ-algebras) are finitary varieties. Note that in a λ-clone we have the derived unary operations $(\boldsymbol{\lambda} x_1), (\boldsymbol{\lambda} x_2), (\boldsymbol{\lambda} x_3), ...$ Thus the classical λ-terms can be interpreted in any λ-clone. In fact, the initial λ-clone Λ is precisely the quotient of classical λ-terms modulo α-conversion.

Lemma 58 *Suppose* $a, b \in \Lambda$.
1. $FV(x) = \{x\}$.
2. $FV(\boldsymbol{\lambda} x.a) = FV(a) - \{x\}$.
3. $FV(ab) = FV(a) \cup FV(b)$.

Example 5.2 *Suppose* $\mathcal{A}$ *is a* λ*-clone.*
1. $\boldsymbol{\lambda} y.y = \lambda((y^+)[x_1/y^+]) = \lambda x_1$.
2. $\boldsymbol{\lambda} y.y = \boldsymbol{\lambda} z.z$ *for any variables* y *and* z.
3. If $\mathcal{A}$ *is a* λ_β*-clone then* $(\boldsymbol{\lambda} y.y)b = b$ *for any* $b \in A$.
4. If $i > 0$ *then* λx_i *has a rank* $i-1$, *and* $\lambda^i x_i$ *is closed.*

Remark 59 *If* $\mathcal{A}$ *is a* λ_β*-clone then* $(\boldsymbol{\lambda} y.a)b = a[b/y]$. *By induction one can see that this rule is sufficient for the calculations in* $\Lambda(\mathcal{A})$ *using the derived binding operations* $\{\boldsymbol{\lambda} y\}$ *and simple substitutions* $\{[b/y]\}$.

A λ-clone can also be defined directly without referring to the unary map λ:

Definition 60 *(Alternate Definition) A* λ*-clone is a clone with the following maps:*
(i) $\boldsymbol{\lambda} x_1 : \mathcal{A} \to \mathcal{A}$ *such that* $(\boldsymbol{\lambda} x_1.a)[a_1, a_2, ...] = (\boldsymbol{\lambda} x_1.(a[x_1, a_2^+, a_3^+, ...]))^-$.
(ii) $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ *is a homomorphism of right* $\mathcal{A}$*-algebras (called the application).*
A λ_β*-clone is a* λ*-clone satisfying the following axiom for any* $a \in A$
(a) $(\boldsymbol{\lambda} x_1.a)x_1 = a$ *(*β*-conversion).*
A $\lambda_{\beta\eta}$*-clone is a* λ_β*-clone satisfying the following axiom for any* $a \in A$
(b) $\boldsymbol{\lambda} x_1.(a^+ x_1) = a^+$ *(*η*-conversion).*

6 Clones in Predicate Logic

In this section we present an ad hoc approach to first-order logic to show how to eliminate the problems related to the complicated notion of variable substitutions encountered in most traditional approaches. We choose E. Mendelson [13] as our main reference. All definitions are given in traditional fashion. Although a few properties are stated in terms of clones, it is straightforward to replace them by checking the properties directly. Historically A. Tarski [19] seems to be the first to address these problems.

Let $\tau_p = \{\mathsf{F}, \Rightarrow\}$ be a type of algebras, where F is a 0-ary operation and $\Rightarrow$ is a binary operation. Any τ_p-algebra is called a *proposition algebra*. For instance, $2 = \{0, 1\}$ is a proposition algebra with $\mathsf{F} = 0$ and $(0 \Rightarrow 0) = (0 \Rightarrow 1) = (1 \Rightarrow 1) = 1$, $(1 \Rightarrow 0) = 0$. If P is a proposition algebra we introduce some further operations: $\neg p = (p \Rightarrow \mathsf{F})$, $\mathsf{T} = \neg\mathsf{F}$, $p \vee q = (\neg p) \Rightarrow q$, $p \wedge q = \neg(p \Rightarrow \neg q)$, $p \Leftrightarrow q = (p \Rightarrow q) \wedge (q \Rightarrow p)$.

Definition 61 *A first-order language $\mathcal{L}$ consisting of*
(i) The set of individual variables $\{x_1, x_2, ...\}$.
(ii) The set $\{\mathbb{F}_n\}_{n\in\mathbb{N}}$ of function symbols.
(iii) The set $\{\mathbb{R}_n\}_{n\in\mathbb{N}}$ of predicate symbols. We assume $\mathbb{R}_2$ contains an element $\approx\, \in \mathbb{R}_2$, called the identity.
(iv) The set of logic symbols $\Rightarrow$, F, and $\forall$.

Let $T(\mathcal{L})$ be the set of terms which is defined inductively as the smallest set such that
(i) $\{x_1, x_2, ...\} \subset T(\mathcal{L})$.
(ii) if $t_1, t_2, ..., t_n$ are terms and $f \in F_n$ then $f(t_1, t_2, ..., t_n)$ is a term.

Let $F_a(\mathcal{L})$ be the set of expressions (called atomic formulas) $r(t_1, t_2, ..., t_n)$ where $r \in \mathbb{R}_n$ and $t_1, t_2, ..., t_n \in T(\mathcal{L})$ (thus $\approx (t_1, t_2) \in F_a(\mathcal{L})$, which will be denoted as $t_1 \approx t_2$).

Let $F(\mathcal{L})$ be the set which is defined inductively as the smallest set such that
(i) $F_a(\mathcal{L}) \subset F(\mathcal{L})$.
(ii) $\mathsf{F} \in F(\mathcal{L})$.
(iii) if $p, q \in F(\mathcal{L})$ then $p \Rightarrow q \in F(\mathcal{L})$.
(iv) If $p \in F(\mathcal{L})$ then $\forall p \in F(\mathcal{L})$.
An element in $F(\mathcal{L})$ is called a *formula* of $\mathcal{L}$.

We define a multiplication $T(\mathcal{L}) \times T(\mathcal{L})^{\mathbb{N}} \to T(\mathcal{L})$ inductively:
a. $x_i[t_1, t_2, ...] = t_i$ for any $t_1, t_2, ... \in T(\mathcal{L})$.
b. $f(s_1, s_2, ..., s_n)[t_1, t_2, ...] = f(s_1[t_1, t_2, ...], s_2[t_1, t_2, ...], ..., s_n[t_1, t_2, ...])$.
$T(\mathcal{L})$ is a clone with the unit $[x_1, x_2, ...]$.

Next we define a multiplication $F(\mathcal{L}) \times T(\mathcal{L})^{\mathbb{N}} \to F(\mathcal{L})$ inductively:
(i) $r(p_1, p_2, ..., p_n)[t_1, t_2, ...] = r(p_1[t_1, t_2, ...], p_2[t_1, t_2, ...], ...)$.
(ii) $\mathsf{F}[t_1, t_2, ...] = \mathsf{F}$.
(iii) $(p \Rightarrow q)[t_1, t_2, ...] = (p[t_1, t_2, ...]) \Rightarrow p[t_1, t_2, ...])$.
(iv) $(\forall p)[t_1, t_2, ...] = \forall(p[x_1, t_1^+, t_2^+, ...]$, where $t_i^+ = t_i[x_2, x_3, ...]$.
Then $F(\mathcal{L})$ is a right $T(\mathcal{L})^{\mathbb{N}}$-act. So $F(\mathcal{L})$ is a locally finitary right $T(\mathcal{L})$-algebra.

Definition 62 *Let $\mathcal{L}$ be a first order language. An interpretation (or a model) M of $\mathcal{L}$ consists of the following ingredients:*
(i) A non-empty set D, called the domain of the interpretation.
(ii) For each function symbol $f \in \mathbb{F}_n$ an assignment of an n-place operation f^M in D, i.e. a function from D^n to D.
(iii) For each predicate symbol $r \in \mathbb{R}_n$ an assignment of an n-place relation r^M in D, i.e. a subset of D^n. We assume $\approx^M = \{(d, d) | d \in D\}$.

Suppose M is an interpretation of $\mathcal{L}$.
We first define a multiplication $T(\mathcal{L}) \times D^{\mathbb{N}} \to D$ inductively:
a. $x_i[d_1, d_2, ...] = d_i$ for any $d_1, d_2, ... \in D^{\mathbb{N}}$.
b. $f(t_1, t_2, ..., t_n)[d_1, d_2, ...] = f^M(t_1[d_1, d_2, ...], t_2[d_1, d_2, ...], ..., t_n[d_1, d_2, ...])$.
Then D is a left $T(\mathcal{L})$-algebra.

Let $2 = \{0, 1\}$. We define a multiplication $F(\mathcal{L}) \times D^{\mathbb{N}} \to 2$ inductively:
a. $r(t_1, t_2, ..., t_n)\tilde{d} = 1$ if and only if $(t_1\tilde{d}, t_2\tilde{d}, ..., t_n\tilde{d}) \in r^M$ for any $\tilde{d} \in D^{\mathbb{N}}$
b. $\mathsf{F}\tilde{d} = 0$.
c. $(p \Rightarrow q)\tilde{d} = 1$ if and only if $p\tilde{d} = 0$ or $q\tilde{d} = 1$.
d. $(\forall p)[d_1, d_2, ...] = 1$ if and only if $p(d, d_1, d_2, ...) = 1$ for any $d \in D$.

Definition 63 *Let M be an interpretation of $\mathcal{L}$.*
1. A sequence $(d_1, d_2, ...)$ in $D^{\mathbb{N}}$ satisfies a formula p iff $p[d_1, d_2, ...] = 1$.
2. A formula p is true for the interpretation M (written $\models_M p$) iff every sequence in $D^{\mathbb{N}}$ satisfies p.
3. p is said to be false for M iff no sequence in $D^{\mathbb{N}}$ satisfies p.
4. M is said to be a model for a set Γ of formulas if and only if every p in Γ is true for M.

Definition 64 *1. A formula p is said to be logically valid iff p is true for every interpretation of $\mathcal{L}$.*
2. p is said to be logically imply q iff in every interpretation, every sequence that satisfies p also satisfies q.
3. p is said to be a logical consequence of a set Γ of formulas iff for every interpretation, every sequence that satisfies every formula in Γ also satisfies p.

If $p \in F(\mathcal{L})$ and $a \in T(\mathcal{L})$ let $p[a/x_i] = p[x_1, x_2, ..., x_{i-1}, a, x_{i+1}, ...]$. In the following we assume $y, z, w, ... \in \{x_1, x_2, ...\}$, which are called *(syntactical)* *variables*. We say a variable y is free for $p \in F(\mathcal{L})$ if $a \neq a[y^+/y]$; otherwise we say that p is independent of y. Denote by $FV(p)$ the set of free variables of p, which is always a finite set. If $FV(p) = \emptyset$ the we say that p is a sentence (or that p is closed, or p has rank 0). We say a formula p has a rank $n > 0$ if $p[[x_1, x_2, .., x_n]] = p$. If p has hank $n \geq 0$ then $FV(p) \subseteq \{x_1, x_2, ..., x_n\}$.

Suppose $\mathcal{L}$ is any first-order language. We introduce some further operations: $y \approx z = \approx [y, z, x_3, x_4, ...]$, $\exists p = \neg\forall\neg p$.

For any variable x_i we derive two maps $\forall x_i, \exists x_i : P \to P$ by

$$\forall x_i.p = \forall(p[x_2, x_3, ..., x_{i-1}, x_i, x_1, x_{i+2}, ...]$$

$$\exists x_i.p = \exists(p[x_2, x_3, ..., x_{i-1}, x_i, x_1, x_{i+2}, ...]$$

Note that $\forall x_1.p = (\forall p)[x_2, x_3, ...]$. So we have $\forall p = (\forall x_1.p)[x_1, x_1, x_2, ...]$, and

$$\forall x_i.p = (\forall x_1.(p[x_2, x_3, ..., x_{i-1}, x_i, x_1, x_{i+2}, ...]))[x_1, x_1, x_2, ...].$$

hence $\forall$ and $\forall x_1$ determines each other, and each $\forall x_i$ can be derived from $\forall x_1$.

Example 6.1 *Suppose M is an interpretation for $\mathcal{L}$. For any formula p we have $(\forall x_i.p)\tilde{d} = \forall(p[x_2, x_3, ..., x_i, x_1, x_{i+2}, ...])\tilde{d} = 1$ for any sequence $\tilde{d}$ if and only if*

$$(p[x_2, x_3, ..., x_i, x_1, x_{i+2}, ...])[d, d_1, d_2, ...] = p([d_1, d_2, ..., d_{i-1}, d, d_{i+1}, ...] = 1$$

for all $d \in D$, which is the classical definition for $\forall x_i.p$.

Lemma 65 *1. If p has rank $n > 0$ then $\forall p$ has rank $n - 1$.*
2. If p is a sentence then $\forall p$ and $\forall x_i.p$ are sentences.
3. If p has rank $n > 0$ then $\forall^n p$ is a sentence.
4. $\forall x_i.p$ is independent of x_i.
5. If p is independent of y then $\forall z.b = \forall y.(b[y/z])$.
6.. If p is independent of y then $\forall y.b = \forall(b^+)$.
7. If $y \neq z$ and $a \in T(\mathcal{L})$ is independent of z then $(\forall z.b)[a/y] = \forall z.(b[a/y])$.

Now it is easy to translate the classical treatment of first order theory into our setting and prove all the fundamental theorems of first order theory.

7 Predicate Algebras

Definition 66 *Let $\mathcal{A}$ be a clone. A predictive algebra with terms in $\mathcal{A}$ is a right $\mathcal{A}$-algebra P together with three homomorphisms of right $\mathcal{A}$-algebras:*

1. $\Rightarrow: P^2 \to P$.
2. $\mathsf{F} : P^0 \to P$ *(i.e.* F *is an element of* P *of rank* 0*.)*
3. $\forall : P^{\mathcal{A}} \to P$.
We also assume that P *has identity*, which is an element $\approx \in P$ of rank 2.

Thus a predicate algebra with terms in $\mathcal{A}$ is an $(\mathcal{A}, \tau_p)$-algebra with a homomorphism $\forall : P^{\mathcal{A}} \to P$ and an element $\approx \in P$ of rank 2. Any proposition algebra may be viewed trivially as a predicate algebra with terms in $\mathcal{A}$ such that all elements are closed and $\approx = \mathsf{T}$.

Example 7.1 *Let* D *be a left* $\mathcal{A}$*-algebra. Since* $2 = \{0, 1\}$ *is a proposition algebra,* $2^{D^{\mathbb{N}}}$ *is an* $(\mathcal{A}, \tau_p)$*-algebra. For* $p \in 2^{D^{\mathbb{N}}}$ *let* $\forall p \in 2^{D^{\mathbb{N}}}$ *such that* $\forall p[d_1, d_2, ...] = 1$ *if and only if* $p[d, d_1, d_2, ...] = 1$ *for all* $d \in D$*. Define* $\approx \in 2^{D^{\mathbb{N}}}$ *such that* $\approx [d_1, d_2, ...] = 1$ *if and only if* $d_1 = d_2$*. Then* $2^{D^{\mathbb{N}}}$ *is a predicate algebra with terms in* $\mathcal{A}$*, called the predicate set algebra for* D*.*

Example 7.2 *Let* B *be a right* $\mathcal{A}$*-algebra. Let* P_B *be the set which is defined inductively as the smallest set such that (a)* $B \subset P_B$*. (b)* $\mathsf{F} \in P_B$*. (c) if* $p, q \in P_B$ *then* $p \Rightarrow q \in P_B$ *and* $\forall p \in P_B$*. Then* P_B *is a predicate algebra with terms in* A*, called the free predicate algebra over* B. In particular, if S is any set then $S \times \mathcal{A}$ is the free right $\mathcal{A}$-algebra over S, and $P_{S \times \mathcal{A}}$ is the *free predicate algebra over* S. Note that if B is locally finitary then so is P_B.

Example 7.3 *The initial predicate algebra with terms in a clone* $\mathcal{A}$ *is a locally finitary predicate algebra* $Eq(\mathcal{A})$*, called the equational logic* for $\mathcal{A}$.

Let P be a predicate algebra with terms in $\mathcal{A}$. For any variable x_i we introduce a new map $\forall x_i : P \to P$ by $\forall x_i.p = \forall(p[x_2, x_3, ..., x_{i-1}, x_i, x_1, x_{i+2}, ...]$. Note that $\forall x_1.p = (\forall p)^+$. So we have $\forall p = (\forall x_1.p)^-$, and

$$\forall x_i.p = (\forall x_1.(p[x_2, x_3, ..., x_{i-1}, x_i, x_1, x_{i+2}, ...])^-.$$

Since $\forall$ and $\forall x_i$ can be derived from $\forall x_1$, a predicate algebra can also be defined using $\forall x_1$ (instead of $\forall$) as the basic operation. Let $\exists p = \neg\forall\neg p$ and $\exists x_i.p = \neg\forall x_i.\neg p$.

Example 7.4 *Suppose* D *is a left* $\mathcal{A}$*-algebra. For any* $p \in 2^{D^{\mathbb{N}}}$ *and any* x_i *we have* $(\forall x_i.p)\tilde{d} = \forall(p[x_2, x_3, ..., x_i, x_1, x_{i+2}, ...])\tilde{d} = 1$ *for any* $\tilde{d} \in D^{\mathbb{N}}$ *if and only if* $p[x_2, x_3, ..., x_i, x_1, x_{i+2}, ...][d, d_1, d_2, ...] = p([d_1, d_2, ..., d_{i-1}, d, d_{i+1}, ...]) = 1$ *for all* $d \in D$*.*

An *interpretation of* P (or a P*-structure*, or a *model of* P) is a pair (D, μ) consisting of a left $\mathcal{A}$-algebra D and a homomorphism $\mu : P \to P(D^{\mathbb{N}})$ of predicate algebras. We say $p \in P$ is *logical valid* (written $\models p$) if for any interpretation (D, μ) we have $\mu(p) = \mathsf{T}$. If $p, q \in P$ then we say that p and q are *logically equivalent* (written $p \equiv q$) if $p \Leftrightarrow q$ is logically valid. The relation

$\equiv$ is a congruence relation on P. The set of congruence classes of P with respect to $\equiv$ is a predictive algebra called the *Lindenbaum-Tarski algebra of* P, denote by $LT(P)$.

Definition 67 *A predicate algebra with terms in a clone $\mathcal{A}$ is called a quantifier algebra with terms in $\mathcal{A}$* if $p \equiv q$ implies that $p = q$, i.e. $P = LT(P)$.

Any predicate set algebra is a quantifier algebra. By definition the class of quantifier algebras is the variety generated by all predicate set algebras. A quantifier algebra is a Boolean algebra with respect to the operations $\vee, \wedge, \neg$ such that each $a \in \mathcal{A}$ induces an endomorphism of this Boolean algebra. We also have existential quantifiers $(\exists x_1), (\exists x_2), ...,$. These data determined a polyadic algebra with terms in $\mathcal{A}$.

Theorem 68 *(cf. [16]) A locally finitary predicate algebra P with terms in a clone $\mathcal{A}$ is a quantifier algebra if and only if the following conditions are satisfied for all $a, b \in P$ and variables y, z:*
1. P is a Boolean algebra with respect to $\vee, \wedge, \neg, \mathsf{F}, \mathsf{T}$.
2. $\exists(a \vee b) = \exists a \vee \exists b$.
3. $a \leq (\exists a)^+$.
4. $x_1 \approx x_1 = \mathsf{T}$.
5. $a \wedge (y \approx z) \leq a[z/y]$, where $a[z/y] = a[x_1, x_2, ..., y^-, z, y^+, ...]$.

Example 7.5 *Let $\mathcal{L}$ be a first-order language. Then*
1. $T(\mathcal{L})$ is a locally finitary clone.
2. $F(\mathcal{L})$ is a locally finitary predicate algebra with terms in $T(\mathcal{L})$.
3. If M is an interpretation of $\mathcal{L}$ then the multiplication $T(\mathcal{L}) \times D^{\mathbb{N}} \to D$ turns D into a left $T(\mathcal{L})$-algebra, and the multiplication $F(\mathcal{L}) \times D^{\mathbb{N}} \to 2$ induces a homomorphism of predicate algebras $F(\mathcal{L}) \to 2^{D^{\mathbb{N}}}$.
4. Conversely if D is a left $T(\mathcal{L})$-algebra then any homomorphism $\mu : F(\mathcal{L}) \to 2^{D^{\mathbb{N}}}$ induces an interpretation M for $\mathcal{L}$ such that $f^M = f(x_1, x_2, ..., x_n) \in D$ for any $f \in \mathbb{F}_n$ and for any $r \in \mathbb{R}_n$ we have $(d_1, d_2, ..., d_N) \in r^M$ iff

$$\mu(r(x_1, x_2, ..., x_n))(d_1, d_2, ..., d_n) = 1.$$

References

[1] P. Aczel, *Notes on the simply typed lambda calculus*, In the Proceedings of the 1997 Computational Logic Advanced Study Institute International Summer School at Marktoberdorf, (1999).

[2] J. Adamek, F. W. Lawvere, J. Rosicky, *On the duality between varieties and algebraic theories*, Algebra Universalis, Basel, Birkhauser, Switzerland. ISSN 0002-5240, vol. 49 (2003), no. 1, 35-49.

[3] H.P. Barendregt, *The Lambda Calculus: Its Seyntax and Semantics*, 2nd edition, North Holland, (1984),

[4] P. M. Cohn, *Universal Algebra*, Kluwer Academic Publishers (1981)

[5] K. Denecke, S L Wismath *Hyperidentities and Clones (Algebra, Logic and Applications Series Volume 14* CRC 2000

[6] M. Fiore, G. Plotkin, D. Turi, *Abstract Syntax and Variable Binding* Proceedings of the 14th Annual IEEE Symposium on Logic in Computer Science, LICS '99, (1999) 193–202

[7] F. E. Linton, *An outline of functorial semantics*, Lecture Notes in Math., Vol **80** Springer-Verlag (1969), 7-52.

[8] Z. Luo, *Clones and Genoids in Lambda Calculus and First Order Logic*, preprint, arXiv:0712.3088v2.

[9] Z. Luo, *Clones and Genoids*, http://www.algebraic.net/cag/.

[10] S. Mac Lane, I. Moerdijk, *Sheaves in Geometry and Logic*, Springer, (1992)

[11] E. Manes, *Algebraic Theories*, Springer-Verlag, 1976.

[12] R. Mckenzie *An algebraic version of categorical equivalence for varieties and more general algebraic categories* Logic and Algebra (Proceedings of the Magari Conference, Siena), Marcel Dekker, New York, (1996) 211-243

[13] E. Mendelson, *Introduction to Mahtematical Logic* , 4th edition, Chapman & Hall/CRC, (1997).

[14] W. D. Neumann, *Representing varieties of algebras by algebras*, J. Austral. Math. Soc. **11** (1970), 1–8.

[15] B. Pareigis and H. Rohrl, *Left linear theories – A generalization of module theory*, Applied Categorical Structures. Vol. **11**, No. 2, (1994), 145-171.

[16] C, C. Pinter, *A Simple Algebra of first Order Logic*, Notre Dame Journal of Formal Logic, Vol. **XIV**, No. 3, (1973), 361-366.

[17] J. Power, *Countable Lawvere Theories and Computational Effects*, Electronic Notes in Theoretical Computer Science, Volume 161, (2006) 59-71

[18] B. M. Schein, V. S. Trohimenko, *Algebras of Multiplace Functions*, Semi-group Forum **17**, (1979), 1-64

[19] A. Tarski, *A simplified formalization of the predicate logic with identity*, Arch. f. Math. Logik. u. Grundl, **7**, 1965 61-79.